THE ARCHIMEDEAN HEIGHT PAIRING FOR DIFFERENTIAL FORMS ON DEGENERATION OF RIEMANN SURFACES

JUNYU CAO

Abstract. We define the Archimedean height pairing for fiberwise cohomologically trivial differential forms on a one-parameter degeneration of Riemann surfaces, and we study its asymptotic behavior. The proof relies on recent work by Dai–Yoshikawa on the asymptotics of small eigenvalues. As an application, we relate this pairing to the current-valued pairing of Filip–Tosatti, extending their construction to broader geometric settings.

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Introduction

Let $\pi\colon X \to S \simeq \mathbb{D}$ be a proper surjective holomorphic map from a complex surface X to a Riemann surface isomorphic to the unit disk. We assume that π has connected fibers and $X_0 := \pi^{-1}(0)$ is the unique singular fiber. Assume that X is equipped with a Kähler metric.

For two smooth real $(1,1)$-forms α, β on X that are cohomologically trivial on smooth fibers $X_s := \pi^{-1}(s)(s \neq 0)$, i.e., $\int_{X_s} \alpha = \int_{X_s} \beta = 0$, we define *the Archimedean height pairing* between them.

Definition 0.1 (Archimedean height pairing, Definition 4.1)**.** *For any $s \neq 0$, let $-\varphi_s \in C^\infty(X_s)$ be a potential for α on X_s, i.e., $\mathrm{dd}^c\varphi_s + \alpha \mid_{X_s} = 0$. Then* the Archimedean height pairing *$\langle \alpha, \beta \rangle$ between α, β is a function on smooth locus $S^\circ := S \setminus \{0\}$:*

$$\langle \alpha, \beta \rangle(s) := \int_{X_s} \varphi_s \beta, \quad s \neq 0.$$

Since π is a submersion on S°, $\langle \alpha, \beta \rangle \in C^\infty(S^\circ)$.

We study its asymptotics around the singular fiber of the degeneration.

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Theorem 0.2 (Theorem 4.5, Corollary 4.6)**.** *There is a constant $c_{\alpha,\beta} \in \mathbb{R}$ such that $\langle \alpha, \beta \rangle - c_{\alpha,\beta} \log |s|^2$ extends continuously to S. In particular, $\langle \alpha, \beta \rangle \in L^\infty(S^\circ)$ if and only if $\langle \alpha, \beta \rangle \in C^0(S)$.*

When X_0 is a reduced divisor and $C_1, \cdots, C_N$ are its irreducible components, the constant $c_{\alpha,\beta}$ is determined in the following way: Let M be the intersection matrix of irreducible components of X_0, given by $(M_{ij})_{1\le i,j\le N} = (C_i \cdot C_j)_{1\le i,j\le N}$, and let M^+ be the Moore-Penrose pseudoinverse of M. Then $c_{\alpha,\beta} = \mathbf{v}_\alpha^T (M^+)^T \mathbf{v}_\beta$, where

$$\mathbf{v}_\alpha = \left(\int_{C_1} \alpha, \dots, \int_{C_N} \alpha \right)^T, \quad \mathbf{v}_\beta = \left(\int_{C_1} \beta, \dots, \int_{C_N} \beta \right)^T.$$

Remark 0.3. The name *Archimedean height pairing* comes from Arakelov theory, e.g. in Bost's work [Bos90]. But their pairings are defined for algebraic cycles/divisors.

Precisely, let D, E be two divisors on the one-parameter degeneration $\pi\colon X \to S$ of algebraic curves. Suppose D, E are of relative degree 0 and their supports do not intersect. For a divisor D, we define its Green's current g_{D_s} on general fibers X_s following [HdJ15, Equation (2.1)]. The g_{D_s} is a distribution on X_s solving the equation

$$\partial\bar{\partial} g_{D_s} + \pi i \delta_{D_s} = 0,$$

where D_s is the restriction of D on X_s. We remark that g_{D_s} is smooth on X_s away from the support of D_s. Then the Archimedean height pairing between D and E is a well-defined function on S°:

$$s \mapsto \langle D_s, E_s \rangle_\infty := g_{D_s}([E_s]) = \sum_i a_i g_{D_s}(q_i),$$

where $E_s = \sum_i a_i q_i$.

In Holmes–de Jong's paper [HdJ15, Theorem 2.1], they derive a similar asymptotic result for the Archimedean height pairing $\langle D_s, E_s \rangle_\infty$ on degenerations of algebraic curves.

Remark 0.4. Similar asymptotic results were obtained by Yoshikawa [Yos06] and Eriksson–Freixas–Mourougane [EFiMM18] in their study of Quillen metrics around singular fibers. Analogous behavior for the canonical heights of elliptic curves around singular fibers was established by Silverman [Sil92, Sil94a, Sil94b] and recently by DeMarco–Mavraki [DM20].

The study of the Archimedean height pairing for forms is motivated by the following dynamical scenario. Let X be a projective elliptic K3 surface with a parabolic automorphism T preserving the fibration:

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "\pi"'] & X \arrow[dl, "\pi"] \\ B := \mathbb{P}^1 & \end{tikzcd}$$

In Filip–Tosatti's paper [FT23, Theorem 3.2.14], they introduced the "*current-valued pairing*" $\eta_B(T, [\alpha])$, which is a current on B, for α a smooth closed real $(1,1)$-form on X that is cohomologically trivial on smooth fibers.

The *current-valued pairing* can be used to characterize the limit properties of large iterations of the automorphism T. For example, they proved in [FT23, Corollary 3.2.20] that

$$\lim_{n\to\infty} \frac{T^n_*\omega}{n^2} = \frac{1}{2}\pi^*\eta_B(T,[T_*\omega-\omega]), \tag{0.1}$$

where ω is a Kähler form on X.

When the singular fibers of π are reduced and irreducible, Filip–Tosatti established the continuity of potentials of η_B in [FT23, Proposition 3.2.11]. In this paper, we remove their assumption on the singular fibers and prove the continuity in general.

Proposition 0.5 (Proposition 5.4)**.** *Let* $\mathrm{Aut}_\pi(X)$ *be the group of automorphisms of* X *preserving the fibration. There exists a finite index subgroup* G *of* $\mathrm{Aut}_\pi(X)$ *such that the* current-valued pairing η_B *restricted on* G *has a continuous potential.*

This proposition is proved by the following observation, which relates *the Archimedean height pairing* and the potential of *current-valued pairing*.

We use u to denote a potential of the *current-valued pairing* $\eta_B(T,[\alpha])$, then we have (in Equation (5.4))

$$u(s) = \int_{X_s} f\omega + \langle \alpha, (T^* - I)\omega\rangle,$$

where $f \in C^\infty(X)$ and ω is a normalized Kähler form on X. The first term $s \mapsto \int_{X_s} f\omega$ is well-studied by Barlet (Theorem 1.3) and is Hölder continuous. Therefore, Proposition 0.5 follows from the continuity of *the Archimedean height pairing*, which arises as the second term of the potential u.

Using Equation (0.1) ([FT23, Corollary 3.2.20]), we have the following result on parabolic dynamics of elliptic K3 surfaces.

Corollary 0.6 (Corollary 5.5)**.** *Let* G *be the finite index subgroup of* $\mathrm{Aut}_\pi(X)$ *defined in Proposition 0.5. Then, for any Kähler form* ω *on* X *and any* $T \in G$*, the limit current* $\eta := \lim_{n\to\infty} \frac{T^n_*\omega}{n^2}$ *has a continuous potential.*

Furthermore, we prove that the convergence of currents $\frac{(T^n)_*\omega}{n^2} \to \frac{1}{2}\pi^*\eta_B(T,[T_*\omega-\omega])$ *cannot be in* $C^0_{\mathrm{loc}}(X\backslash X_0)$*. Therefore, the sequence* $\left\{\frac{T^n_*\omega}{n^2}\right\}_{n\geq 1}$ *provides a counterexample to Tosatti's question* [Tos25, Question 1.5] *when we let* ω *be a Ricci-flat metric.*

Remark 0.7. The regularity of limit currents is also discussed in Cantat–Dujardin's recent paper [CD23b, Theorem D].

If the automorphism is hyperbolic, its associated limit current will generally have a Hölder continuous potential. However, it is unknown whether the potential of a limit current given by a parabolic automorphism is Hölder continuous.

Remark 0.8. Other related works on the parabolic dynamics of elliptic surfaces with general singular fibers include Duistermaat's monograph [Dui10] and Cantat–Dujardin's recent paper [CD23a, Section 3].

Here we sketch the proof of Theorem 0.2. By a standard reduction, we can assume that X_0 is reduced and α, β have a zero integral on each irreducible component of X_0. Then it remains to show the continuity of the pairing in Theorem 4.4: We introduce *the preferred potentials* (Definition 3.3) for α, β. Then by Dai–Yoshikawa's work [DY25], we are able to derive some estimates on preferred potentials (Theorem 3.4, Theorem 3.5). Together with some continuity results of fiber integral around singular fibers (Proposition 1.13, Corollary 1.14), we derive the continuity of the pairing (which is a fiber integral) in Theorem 4.4.

This article is organized as follows. In Section 1, we collect and prove some technical results. In Section 2, we study the properties of eigenfunctions with small eigenvalues on smooth fibers of a degeneration. We use Dai-Yoshikawa's work [DY25] heavily in this section. In Section 3, we introduce the notion of a *preferred potential* and establish its asymptotic estimates based on Section 2. In Section 4, we introduce the *Archimedean height pairing* and prove its asymptotics around singular fibers. In Section 5, we apply previous results to study the parabolic dynamics of K3 surfaces. In Appendix A, we prove a technical lemma on the asymptotics of a fiber integral.

Notation and conventions

On a complex manifold X, we define $\mathrm{d}^c = \frac{i}{4\pi}(\overline{\partial} - \partial)$, so $\mathrm{dd}^c = \frac{i}{2\pi}\partial\overline{\partial}$.

On a Kähler manifold (X, ω) with $\omega = i\sum g_{j\bar{k}} dz_j \wedge d\bar{z}_k$, we use the positive-definite analyst's Laplacian $\Delta_\omega f = -\sum g^{j\bar{k}} \frac{\partial^2 f}{\partial z_j \partial \bar{z}_k}$. Under our normalization of dd^c, we have $-\Delta_\omega = 2\pi \operatorname{tr}_\omega \mathrm{dd}^c$.

With this convention, the Poincaré-Lelong equation for a holomorphic section s of a Hermitian line bundle (L, h) is:

$$\mathrm{dd}^c \log |s|_h^2 = [D] - c_1(L, h)$$

where $[D]$ is the current of integration over the vanishing locus of s, and $c_1(L, h)$ is the first Chern form.

For a closed smooth differential form α on a compact differential manifold, we use $[\alpha]$ to denote its de Rham cohomology class.

We use $\mathbb{D} := \{z \in \mathbb{C} \colon \ |z| < 1\}$ to denote the unit disk in complex plane, and $\mathbb{D}^\circ := \mathbb{D} \setminus \{0\}$ the punctured unit disk. For a positive number $r > 0$, we also define $\mathbb{D}_r := \{z \in \mathbb{C} \colon \ |z| < r\}$ and $\mathbb{D}_r^\circ := \mathbb{D}_r \setminus \{0\}$.

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1. Technical Preparation

We collect and prove some technical results that will be used later.

1.A. Laplacian on singular varieties

Let X be an algebraic variety with the induced Fubini-Study metric, and let X_{reg} be the regular part of X.

Following Li-Tian's work [LT95, Section 4], we define the ordinary Laplacian Δ on compactly supported smooth functions on X_{reg} and let $\overline{\Delta}$ be its Friedrichs extension. Then we have

Theorem 1.1 ([LT95, Section 4]). *The Friedrichs extension $\overline{\Delta}$ defined on functions is self-adjoint on a compact variety X (without boundary) whose singular set $\operatorname{Sing} X$ is at least of real codimension 2.*

When $f \in W^{1,2}(X_{\mathrm{reg}})$ is harmonic, *i.e., $\Delta f = 0$ almost everywhere, we have $\mathrm{d}f = 0$ and f is locally constant.*

Remark 1.2. This theorem holds when X is a Kähler space, see [GPSS23, Lemma 11.2].

1.B. Continuity of fiber integral à la Barlet

In [Bar82], Barlet derived the asymptotic expansion of the fiber integral for the degeneration of complex manifolds near the singular fiber.

Theorem 1.3 ([Bar82, Théorème 1]). *Let $D = \{s \in \mathbb{C} \mid |s| < 1\}$, and let X be a reduced and irreducible complex analytic space of dimension $n+1$. Let $\pi : X \to D$ be a surjective holomorphic map, and let us denote by X_s the cycle $\pi^{-1}(s)$. Let K be a compact subset of X. There exist rational numbers $r_1 \leq \cdots \leq r_k$ in $[0, 2) \cap \mathbb{Q}$ such that for every C^∞ differential form φ of type (n, n) on X with support in K, the function $F_\varphi := \int_{X_s} \varphi$ admits, as $s \to 0$, an asymptotic expansion of the form:*

$$F_\varphi(s) \sim \sum_{\substack{r=r_1,\ldots,r_k \\ j=0,\ldots,n \\ (m,m')\in\mathbb{N}^2}} T^{r,j}_{m,m'}(\varphi) s^m \overline{s}^{m'} |s|^r (\log |s|)^j$$

where $T^{r,j}_{m,m'}$ is a $(1,1)$-current on X.

Since $F_\varphi(s)$ does not blow up as $s \to 0$, we have $T^{0,j}_{0,0}(\varphi) = 0$ for $j \geq 1$. Let $0 < r$ be the minimum of non-zero $r_1 \leq \cdots \leq r_k$ in $[0, 2) \cap \mathbb{Q}$, then for $0 < a < \min(r, 1)$, we have

$$|F_\varphi(s) - F_\varphi(0)| \leq C(\varphi)\, |s|^a .$$

Thus $F_\varphi(s)$ is Hölder continuous around 0.

Remark 1.4. When the base of the fibration π is of higher dimension, Takayama extends Barlet's result in his recent paper [Tak20].

1.C. Convergence on the singular fiber

Let $\pi \colon X \to \mathbb{D}$ be a proper surjective holomorphic map from a complex manifold X of complex dimension $p = q+1$ to the unit disk. Suppose that π has connected fiber and $X_0 := \pi^{-1}(0)$ is the unique singular fiber. We define $\operatorname{Sing}(X_0) := \{x \in X_0 \mid d\pi_x = 0\}$ and $X_{0,\mathrm{reg}} := X_0 \setminus \operatorname{Sing}(X_0)$. Let $X_s := \pi^{-1}(s)$.

We introduce the following frequently used notion of convergence on the singular fiber.

Definition 1.5 (Convergence on the singular fiber). *Under the above settings, let $(s_n)_{n\geq1} \subset \mathbb{D}^\circ$ be a sequence such that $s_n \to 0$ as $n \to \infty$. Let $(f_{s_n})_{n\geq1}$ be a family of C^k functions on X_{s_n} and f be a C^k function on $X_{0,\mathrm{reg}}$. Around any $p \in X_{0,\mathrm{reg}}$, the map π is a locally trivial fibration by Ehresmann's theorem, so we have an open neighborhood*

U of p with the following diagram:

$$\begin{tikzcd} i: U \arrow[r, "\simeq"] \arrow[dr, "\pi"'] & V \times \mathbb{D}_\epsilon \arrow[d] \\ & \mathbb{D}_\epsilon \end{tikzcd}$$

Then we have a family of diffeomorphisms $i_s\colon U \cap X_s \to V \times \{s\}$. For $s \in \mathbb{D}_\epsilon$, we define $\widetilde{f_s} := (i_s)_*(f_s)$ to be a C^k function on V, and define $\widetilde{f} := (i_0)_*(f)$.

We say f_{s_n} converges to *f* in *C^k_{loc}* on *$X_{0,\text{reg}}$ if around any $p \in X_{0,\text{reg}}$, we have a coordinate chart $U \simeq V \times \mathbb{D}_\epsilon$ and $\widetilde{f_{s_n}} \to \widetilde{f}$ in the $C^k_{\text{loc}}(V)$ topology. This notion of convergence does not depend on the choice of coordinate charts.*

Remark 1.6. Let $f\colon X \setminus X_0 \to \mathbb{R}$ be a continuous function. Then f can be extended to be a continuous function on $X \setminus \operatorname{Sing} X_0$ if and only if f_{s_n} converges to a continuous function $f_0 \in C^0(X_{0,\text{reg}})$ in C^0_{loc} on $X_{0,\text{reg}}$ for any sequence $s_n \to 0$.

We shall show how the convergence on the singular fiber could imply the convergence of the fiber integral in the following.

Lemma 1.7. *Suppose $f_{s_n} \to f$ in C^0_{loc} on $X_{0,\text{reg}}$ and f is a continuous function on $X_{0,\text{reg}}$. Let ω be a Hermitian form on X. Then*

$$\int_{X_{0,\text{reg}}} |f|^l\, \omega^q \leq \limsup_{n\to\infty} \int_{X_{s_n}} |f_{s_n}|^l\, \omega^q$$

for $1 \leq l < \infty$.

Generally, if $f_{s_n} \to f$ in C^k_{loc} on $X_{0,\text{reg}}$, then

$$\int_{X_{0,\text{reg}}} |\nabla^r f|^l\, \omega^q \leq \limsup_{n\to\infty} \int_{X_{s_n}} |\nabla^r f_{s_n}|^l\, \omega^q$$

for $0 \leq r \leq k, 1 \leq l < \infty$.

Before the proof, we need a partition of unity tailored for our situation:

Lemma 1.8. *Let X be a smooth manifold and $K \subset X$ be a compact subset. Suppose $\{U_i\}_{i=1}^N$ is a finite open covering of K. Then there exists a collection of smooth functions $\{\rho_i\}_{i=1}^N$ with $\rho_i : X \to \mathbb{R}$ such that:*

(1) $\operatorname{supp}(\rho_i) \subset U_i$ for each $i = 1, \dots, N$.
(2) $0 \leq \rho_i(x) \leq 1$ for all $x \in X$.
(3) $\sum_{i=1}^N \rho_i(x) = 1$ for all $x \in K$.
(4) $\sum_{i=1}^N \rho_i(x) \leq 1$ for all $x \in X$.

Proof. (of Lemma 1.8) Since K is compact and $\{U_i\}_{i=1}^N$ is an open covering of K, the collection of open sets $\mathcal{U} = \{U_1, U_2, \cdots, U_N, X \setminus K\}$ is an open covering of X. Let $\{\rho_1, \rho_2, \cdots, \rho_N, \rho_{N+1}\}$ be a partition of unity subordinate to $\mathcal{U}$. Then we obtain the desired $\{\rho_i\}_{i=1}^N$. □

Proof. (of Lemma 1.7) Let $U \simeq V \times \mathbb{D}$ be a coordinate chart around some point in $X_{0,\text{reg}}$ trivializing the fibration $\pi\colon X \to \mathbb{D}$. Let h be a continuous function on X compactly supported on U. Then if $f_{s_n} \to f$ in C^k_{loc} on $X_{0,\mathrm{reg}}$, we have $h_{s_n} |\nabla^r f_{s_n}| \to h_0 |\nabla^r f|$ in the $C^0(V)$-topology. In particular, we have $\int_{X_{0,\mathrm{reg}}} h |\nabla^r f|^l \omega^q = \lim_{n\to\infty} \int_{X_{s_n}} h |\nabla^r f_{s_n}|^l \omega^q$.

For any compact subset K of $X_{0,\mathrm{reg}}$, let $\{U_i\}_{i=1}^N$ be an open covering of K by open sets in $X \setminus \operatorname{Sing} X_0$ and we assume that each $U_i \simeq V_i \times \mathbb{D}$ is a coordinate chart that trivializes the fibration π.

We then take the partition of unity $\{h_i\}_{i=1}^N$ subordinate to $\{U_i\}_{i=1}^N$ as in Lemma 1.8, then for each i, we have

$$\int_{X_{s_n}} h_i |\nabla^r f_{s_n}|^l \omega^q \to \int_{X_{0,\mathrm{reg}}} h_i |\nabla^r f|^l \omega^q.$$

Therefore, we have

$$\begin{aligned}\int_K |\nabla^r f|^l \omega^q \le \sum_{i=1}^N \int_{X_{0,\mathrm{reg}}} h_i |\nabla^r f|^l \omega^q &= \lim_{n\to\infty} \sum_{i=1}^N \int_{X_{s_n}} h_i |\nabla^r f_{s_n}|^l \omega^q \\ &\le \limsup_{n\to\infty} \int_{X_{s_n}} |\nabla^r f_{s_n}|^l \omega^q.\end{aligned}$$

As K can be taken to be any compact subset of $X_{0,\mathrm{reg}}$, we conclude that

$$\int_{X_{0,\mathrm{reg}}} |\nabla^r f|^l \omega^q \le \limsup_{n\to\infty} \int_{X_{s_n}} |\nabla^r f_{s_n}|^l \omega^q.$$

□

If f_{s_n} in the previous lemma (Lemma 1.7) is zero in a tubular neighborhood of $\operatorname{Sing} X_0$, then we have the equality for the limit of fiber integral.

Lemma 1.9. *Suppose f_{s_n} converges to f in C^0_{loc} on $X_{0,\mathrm{reg}}$ and there exists $\epsilon > 0$ such that f_{s_n}, f are zero in a tubular neighborhood $B(\operatorname{Sing} X_0, \epsilon) := \{p \in X : \operatorname{dist}(p, \operatorname{Sing} X_0) < \epsilon\}$ of $\operatorname{Sing} X_0$. Then for any smooth (q,q) form α, we have*

$$\int_{X_{s_n}} f_{s_n}\alpha \to \int_{X_0} f\alpha, \quad \text{as } n \to \infty.$$

Proof. Let $U_\epsilon = B(\operatorname{Sing} X_0, \epsilon)$ be an open tubular neighborhood of the singular locus. Since $X_0 \setminus U_\epsilon$ is compact and contained in the regular locus of X_0, we can cover it by a finite number of open sets $\{U_i\}_{i=1}^N$ in $X \setminus \operatorname{Sing} X_0$ and we assume that each $U_i \simeq V_i \times \mathbb{D}$ is a coordinate chart that trivializes the fibration π. Then we have an open covering $U_\epsilon \cup \{U_i\}_{i=1}^N$ of X_0.

Since π is proper, we can find $\epsilon_0 > 0$ such that $\pi^{-1}(\overline{\mathbb{D}_{\epsilon_0}}) \subset U_\epsilon \cup \cup_{i=1}^N U_i$. We apply Lemma 1.8 to the compact set $\pi^{-1}(\overline{\mathbb{D}_{\epsilon_0}})$ and its open covering $U_\epsilon \cup \{U_i\}_{i=1}^N$, then we have a collection of smooth functions $\{\rho_i : U_i \to \mathbb{R}\}_{i=0}^N$ (we denote U_ϵ by U_0) such that

- $\operatorname{supp}(\rho_i) \subset U_i$ for each $i = 0, 1, \ldots, N$,
- $0 \le \rho_i(x) \le 1$ for all $x \in X$,
- $\sum_{i=0}^N \rho_i(x) = 1$ for all $x \in \pi^{-1}(\overline{\mathbb{D}_{\epsilon_0}})$.

Then for $n \gg 1$ such that $|s_n| < \epsilon_0$, we have

$$\int_{X_{s_n}} f_{s_n}\alpha = \sum_{i=0}^N \int_{X_{s_n}} \rho_i f_{s_n}\alpha = \sum_{i=0}^N \int_{X_{s_n}\cap U_i} \rho_i f_{s_n}\alpha$$

And
$$\int_{X_0} f\alpha = \sum_{i=0}^{N} \int_{X_0} \rho_i f\alpha = \sum_{i=0}^{N} \int_{X_0\cap U_i} \rho_i f\alpha$$
Since f_{s_n}, f are zero in $U_0 = U_\epsilon$, we have
$$\int_{X_0\cap U_i} \rho_0 f\alpha = 0 = \int_{X_{s_n}\cap U_0} \rho_0 f_{s_n}\alpha$$
For $1 \leq i \leq N$, the fibration π on $U_i \simeq V_i \times \mathbb{D}$ is trivial and $(\rho_i f_{s_i}\alpha)\mid_{V_i} \to (\rho_i f\alpha)\mid_{V_i}$ in C^0_{loc}. So
$$\int_{X_{s_n}\cap U_i} \rho_i f_{s_n}\alpha = \int_{V_i} (\rho_i f_{s_i}\alpha)\mid_{V_i} \to \int_{V_i} (\rho_i f\alpha)\mid_{V_i} = \int_{X_0\cap U_i} \rho_i f\alpha.$$
In summary, we have
$$\int_{X_{s_n}} f_{s_n}\alpha \to \int_{X_0} f\alpha, \quad \text{as } n \to \infty.$$
□

We can extend above lemma if the contribution of fiber integral around singularities is small enough, which is the following lemma.

Lemma 1.10. *Suppose f_{s_n} converges to f in C^0_{loc} on $X_{0,\text{reg}}$ and X_0 is reduced. Let ω be a Kähler metric on X.*

For any $\epsilon > 0$, we define
$$I(\epsilon) = \limsup_{n\to\infty} \int_{B(\operatorname{Sing} X_0,\epsilon)\cap X_s} |f_{s_n}|\,\omega^q,$$
where $B(\operatorname{Sing} X_0, \epsilon) := \{p \in X : \operatorname{dist}(p, \operatorname{Sing} X_0) < \epsilon\}$ is a tubular neighborhood of $\operatorname{Sing} X_0$.

If $I(\epsilon) \to 0$ as $\epsilon \to 0$ and f is L^1-integrable on X_0, then for any smooth (q, q) form α, we have
$$\int_{X_{s_n}} f_{s_n}\alpha \to \int_{X_0} f\alpha, \quad \textit{as } n \to \infty.$$

Proof. For any fixed $\epsilon > 0$, we have a cutoff function $\eta_\epsilon \in C^\infty(X)$ such that
- $0 \leq \eta_\epsilon \leq 1$,
- $\eta_\epsilon = 1$ on $B(\operatorname{Sing} X_0, \epsilon/2)$,
- $\eta_\epsilon = 0$ on $X \setminus B(\operatorname{Sing} X_0, \epsilon)$.

Then by Lemma 1.9, we have
$$\int_{X_{s_n}} f_{s_n}(1-\eta_\epsilon)\alpha \to \int_{X_0} f(1-\eta_\epsilon)\alpha, \quad \text{as } n \to \infty.$$
Therefore, we have
$$\limsup_{n\to\infty} \left| \int_{X_{s_n}} f_{s_n}\alpha - \int_{X_0} f\alpha \right| \leq \limsup_{n\to\infty} \left| \int_{X_{s_n}} f_{s_n}\eta_\epsilon\alpha \right| + \left| \int_{X_0} f\eta_\epsilon\alpha \right|.$$
For the first term, we have
$$\left| \int_{X_{s_n}} f_{s_n}\eta_\epsilon\alpha \right| \leq \left\| \frac{\alpha}{\omega^q} \right\|_{L^\infty} \int_{B(\operatorname{Sing} X_0,\epsilon)} |f_{s_n}|\,\omega^q.$$

So

$$\limsup_{n\to\infty}\left|\int_{X_{s_n}} f_{s_n}\eta_\epsilon\alpha\right| \le \left\|\frac{\alpha}{\omega^q}\right\|_{L^\infty} I(\varepsilon).$$

We then have

$$\limsup_{n\to\infty}\left|\int_{X_{s_n}} f_{s_n}\alpha - \int_{X_0} f\alpha\right| \le \left\|\frac{\alpha}{\omega^q}\right\|_{L^\infty} I(\varepsilon) + \left|\int_{X_0} f\eta_\epsilon\alpha\right|. \tag{1.1}$$

Note that

$$\left|\int_{X_0} f\eta_\epsilon\alpha\right| \le \left\|\frac{\alpha}{\omega^q}\right\|_{L^\infty} \int_{B(\operatorname{Sing} X_0,\epsilon)\cap X_0} |f|\,\omega^q\omega^q$$

and $\operatorname{vol}(B(\operatorname{Sing} X_0,\epsilon)\cap X_0) \to 0$ as $\epsilon\to 0$ by Lemma 1.12. So $\left|\int_{X_0} f\eta_\epsilon\alpha\right| \to 0$ as $\epsilon\to 0$ by L^1-integrability of $f\alpha$.

Letting $\epsilon\to 0$ in Equation (1.1), we obtain

$$\limsup_{n\to\infty}\left|\int_{X_{s_n}} f_{s_n}\alpha - \int_{X_0} f\alpha\right| = 0.$$

We then conclude the result. □

We shall see that $I(\epsilon)\to 0$ is natural under certain uniform integrability conditions.

Lemma 1.11. *Suppose f_{s_n} converges to f in C^0_{loc} on $X_{0,\mathrm{reg}}$ and X_0 is reduced. Let ω be a Kähler metric on X. If there is $1<r\le\infty$ and a uniform constant $L<\infty$ such that $\|f_{s_n}\|_{L^r(X_{s_n},\omega_{s_n})}<L$, then for any smooth (q,q) form α, we have*

$$\int_{X_{s_n}} f_{s_n}\alpha \to \int_{X_0} f\alpha, \quad \textit{as } n\to\infty.$$

Proof. By Lemma 1.10, it suffices to check the following two conditions

- The contribution of fiber integral around singularities
$$I(\epsilon) = \limsup_{n\to\infty}\int_{B(\operatorname{Sing} X_0,\epsilon)\cap X_s} |f_{s_n}|\,\omega^q$$
goes to zero as $\epsilon\to 0$.
- The f is L^1-integrable on X_0.

The first condition is justified by the Hölder inequality and the volume estimate of tubular neighborhood in Lemma 1.12

$$\begin{aligned}\limsup_{n\to\infty}\int_{B(\operatorname{Sing} X_0,\epsilon)\cap X_s} |f_{s_n}|\,\omega^q &\le \limsup_{n\to\infty}\left(\int_{X_{s_n}} |f_{s_n}|^r\,\omega^q\right)^{1/r}\left(\int_{B(\operatorname{Sing} X_0,\epsilon)\cap X_{s_n}} \omega^q\right)^{1/p}\\ &\le L\limsup_{n\to\infty}\operatorname{vol}(B(\operatorname{Sing} X_0,\epsilon)\cap X_{s_n})\\ &\le LC\epsilon^2 \quad (\text{by Lemma 1.12}).\end{aligned}$$

The second condition follows from Lemma 1.7, we have

$$\|f\|_{L^r(X_0)} \le \limsup_{n\to\infty}\|f_{s_n}\|_{L^r} < L$$

for $1<r\le\infty$. So $f\in L^1(X_0)$. □

We prove the uniform bound for volume of tubular neighborhoods of singular set, which is used in previous two lemmas and is standard:

Lemma 1.12 (Volume bound for singular tubular neighborhoods)**.** *Let $\pi\colon X \to \mathbb{D}$ be a proper surjective holomorphic map from a complex manifold X of complex dimension $p = q + 1$ to the unit disk. Suppose that π has connected fiber and $X_0 := \pi^{-1}(0)$ is the unique singular fiber. Suppose X_0 is reduced. Then* $\operatorname{Sing} X_0$ *is of complex codimension ≥ 1 in X_0.*

Let ω be a Kähler metric on X, and denote by $B(\operatorname{Sing} X_0, \epsilon)$ the ϵ-tubular neighborhood of $\operatorname{Sing} X_0$ *in X with respect to the distance induced by ω. Then there exist constants $C > 0$ and $\epsilon_0 > 0$ such that for all $s \in \mathbb{D}$ near 0 (including $s = 0$) and all $0 < \epsilon < \epsilon_0$, we have*

$$\int_{B(S,\epsilon)\cap X_s} \omega^q \leq C\epsilon^2.$$

Proof. Since $\operatorname{Sing} X_0$ is a compact complex analytic subvariety of X_0 of complex dimension less than $q-1$, its $2(q-1)$-dimensional Hausdorff measure is finite. Thus, there exists a constant $C_1 > 0$ such that for any sufficiently small $\epsilon > 0$, the compact set $\operatorname{Sing} X_0$ can be covered by a finite number of balls $N(\epsilon)$ of radius ϵ centered at points $x_i \in \operatorname{Sing} X_0$, where

$$N(\epsilon) \leq C_1\epsilon^{-2(q-1)}.$$

The ϵ-tubular neighborhood $B(\operatorname{Sing} X_0, \epsilon)$ is strictly contained in the union of the concentric balls $B(x_i, 2\epsilon)$.

Next, we bound the volume of X_s within these local balls. For any s (including $s = 0$), X_s is a purely q-dimensional complex analytic cycle. Because ω is a Kähler metric on X, the standard Bishop-Lelong theorem (or monotonicity formulas) guarantees that the volume ratio is monotonically increasing. Thus, we take $R_0 > 2\epsilon_0$ such that for all $x \in S$ and all $2\epsilon < 2\epsilon_0 < R_0$, we have

$$\frac{1}{(2\epsilon)^{2q}} \int_{B(x,2\epsilon)\cap X_s} \omega^q \leq \frac{1}{R_0^{2q}} \int_{B(x,R_0)\cap X_s} \omega^q.$$

Furthermore, because $d\omega = 0$ and the fibers X_s are homologous in X, the total volume of the fibers $V = \int_{X_s} \omega^q$ is a topological constant independent of s. We have a uniform constant C_2 and the following inequality

$$\int_{B(x,2\epsilon)\cap X_s} \omega^q \leq \left(\frac{V}{R_0^{2q}}\right)(2\epsilon)^{2q} := C_2\epsilon^{2q}.$$

Finally, combining the covering estimate and the local volume bound, we obtain, by the subadditivity of the volume,

$$\int_{B(\operatorname{Sing} X_0,\epsilon)\cap X_s} \omega^q \leq \sum_{i=1}^{N(\epsilon)} \int_{B(x_i,2\epsilon)\cap X_s} \omega^q \leq N(\epsilon)\cdot C_2\epsilon^{2q} \leq C_1C_2\epsilon^2.$$

Setting $C = C_1C_2$, the bound holds uniformly for all s near 0, completing the proof. $\square$

Finally, we give a sufficient condition on the existence of limit of convergence on the singular fiber:

Proposition 1.13. *We work under the above settings and assume that X_0 is reduced. Let φ be a real function on $X \setminus X_0$. We assume that $\varphi|_{X_s}$ is smooth on all regular fibers X_s for $s \neq 0$, and define a family of smooth $(1,1)$ forms $\beta_s := \mathrm{dd}^c(\varphi|_{X_s})$ on each X_s for $s \neq 0$.*

Let ω be a Kähler form on X. If the following conditions hold:

- *Around any point of $X_{0,\mathrm{reg}}$, we have an open neighborhood U such that the L^∞-norms $\|\mathrm{tr}_{\omega_s}\beta_s\|_{L^\infty(X_s \cap U)}$ are uniformly bounded for $|s| \ll 1$.*
- *The family of functions $f_s := \mathrm{tr}_{\omega_s}\beta_s$ converges in C^0_{loc} to a continuous function f_0 on $X_{0,\mathrm{reg}}$.*
- *The L^2-norms $\|\varphi_s\|_{L^2(X_s,\omega_s)}$ and $\|\mathrm{d}\varphi_s\|_{L^2(X_s,\omega_s)}$ are uniformly bounded for $|s| \ll 1$.*

Then for any sequence $(s_n)_{n\geq 1} \subset \mathbb{D}^\circ$ such that $s_n \to 0$ and any $0 < a < 1$, there exists a subsequence of s_n (by abusing the notation, we denote the subsequence by s_n itself) such that

- *φ_{s_n} converges in $C^{1,a}_{\mathrm{loc}}$ to a $C^{1,a}$ function φ_0 on $X_{0,\mathrm{reg}}$,*
- *$\varphi_0 \in W^{2,p}_{\mathrm{loc}}(X_{0,\mathrm{reg}})$ for any $1 < p < \infty$, and φ_0 solves $-\Delta_{\omega_0}\varphi_0 = 2\pi f_0$ almost everywhere on $X_{0,\mathrm{reg}}$,*
- *$\varphi_0 \in W^{1,2}(X_{0,\mathrm{reg}})$.*
- *If α is a smooth (q,q)-form, then*

$$\int_{X_{s_n}} \varphi_{s_n}\alpha \to \int_{X_0} \varphi_0\alpha$$

Furthermore, for two different limits φ_0, φ_0', their difference $\varphi_0 - \varphi_0'$ is locally constant on $X_{0,\mathrm{reg}}$.

Proof. Around any point of $X_{0,\mathrm{reg}}$, we have an open neighborhood U such that the L^∞-norms $\|\mathrm{tr}_{\omega_s}\beta_s\|_{L^\infty(X_s\cap U)}$ are uniformly bounded. By shrinking U, we assume $U \simeq V \times \mathbb{D}_\epsilon$ that trivializes the fibration π.

Around any point of V, we take a relatively compact coordinate ball W. Then φ_s solves

$$-\Delta_{\omega_s}\varphi_s = 2\pi \mathrm{tr}_{\omega_s}\mathrm{dd}^c_z\varphi_s = 2\pi\mathrm{tr}_{\omega_s}\beta_s \quad \text{on } W.$$

Note that Δ_{ω_s} is a family of uniformly elliptic operators for $|s| \ll 1$. By standard local L^∞ bounds for elliptic PDEs (see Gilbarg–Trudinger [GT01, Theorem 8.24]), we can shrink W to a slightly smaller domain $\widetilde{W} \Subset W$ to obtain

$$\begin{aligned}\|\varphi_s\|_{L^\infty(\widetilde{W})} &\leq C\left(\|\varphi_s\|_{L^2(W)} + \|\mathrm{tr}_{\omega_s}\beta_s\|_{L^\infty(W)}\right)\\ &\leq C\left(\|\varphi_s\|_{L^2(X_s)} + \|\mathrm{tr}_{\omega_s}\beta_s\|_{L^\infty(X_s\cap U)}\right),\end{aligned}$$

where C is a uniform constant independent of s. By our assumptions, the right-hand side is uniformly bounded in s.

Next, by interior L^p estimates for elliptic equations (see [GT01, Theorem 9.11]), for any $1 < p < \infty$, we can further shrink $\widetilde{W}$ to W' to obtain

$$\begin{aligned}\|\varphi_s\|_{W^{2,p}(W')} &\leq C_p\left(\|\varphi_s\|_{L^\infty(\widetilde{W})} + \|\mathrm{tr}_{\omega_s}\beta_s\|_{L^p(\widetilde{W})}\right)\\ &\leq C_p'\left(\|\varphi_s\|_{L^\infty(\widetilde{W})} + \|\mathrm{tr}_{\omega_s}\beta_s\|_{L^\infty(X_s\cap U)}\right).\end{aligned}$$

Thus, $\|\varphi_s\|_{W^{2,p}(W')}$ is uniformly bounded for any $p > 1$.

By the Sobolev embedding theorem (Morrey's inequality), for $p > 2q$ (where $q = \dim_{\mathbb{C}} X_s$), $W^{2,p}(W')$ embeds compactly into $C^{1,b}(W')$ for $0 < b < 1 - 2q/p$. Since p can be chosen arbitrarily large, $\|\varphi_s\|_{C^{1,b}(W')}$ is uniformly bounded for any $0 < b < 1$.

To patch these local estimates together, note that $X_{0,\mathrm{reg}}$ is second-countable, so we can choose a countable exhaustion by compact subsets $K_1 \Subset K_2 \Subset \cdots \Subset X_{0,\mathrm{reg}}$ such that $\cup_m K_m = X_{0,\mathrm{reg}}$. For each K_m, we can cover it by finitely many such charts W'. By the Ascoli-Arzelà theorem and diagonal argument, we can extract a single subsequence (s_n) such that φ_{s_n} converges in $C^{1,a}_{\mathrm{loc}}$ to a limit function $\varphi_0 \in C^{1,a}(X_{0,\mathrm{reg}})$ for any given $a \in (0, 1)$. Furthermore, the uniform bounds in $W^{2,p}_{\mathrm{loc}}$ guarantee that $\varphi_0 \in W^{2,p}_{\mathrm{loc}}(X_{0,\mathrm{reg}})$ and that $\varphi_{s_n} \to \varphi_0$ weakly in $W^{2,p}_{\mathrm{loc}}$.

Since $f_{s_n} := \mathrm{tr}_{\omega_{s_n}} \beta_{s_n}$ converges in C^0_{loc} to the continuous limit f_0, passing to the limit in the PDE implies that φ_0 satisfies $-\Delta_{\omega_0} \varphi_0 = 2\pi f_0$ almost everywhere on $X_{0,\mathrm{reg}}$.

By the third condition and Lemma 1.7, we have $\varphi_0 \in W^{1,2}(X_{0,\mathrm{reg}})$.

Since $\|\varphi_s\|_{L^2(X_s,\omega_s)}$ is uniformly bounded, by Lemma 1.11 we have

$$\int_{X_{s_n}} \varphi_{s_n} \alpha \to \int_{X_0} \varphi_0 \alpha$$

for any smooth (q, q)-form α on X.

Suppose φ_0 and φ_0' are two limits arising from different subsequences of φ_{s_n}. Both solve the same equation $-\Delta_{\omega_0} \varphi = 2\pi f_0$ almost everywhere on $X_{0,\mathrm{reg}}$. So their difference $h := \varphi_0 - \varphi_0' \in W^{2,p}_{\mathrm{loc}}(X_{0,\mathrm{reg}})$ is a weakly harmonic function on $X_{0,\mathrm{reg}}$ with respect to ω_0. Since $h \in W^{1,2}(X_{0,\mathrm{reg}})$ globally and the singular locus of X_0 is of real codimension at least 2, by Theorem 1.1, h is locally constant. □

As a corollary, we have the following result on continuity of fiber integral

Corollary 1.14. *We work under the assumption and notations of Proposition 1.13. Assume that all conditions in Proposition 1.13 are satisfied.*

If α is a smooth (q, q)-form such that $\int_C \alpha = 0$ for each irreducible component C of X_0, then the fiber integral $I(s) = \int_{X_s} \varphi_s \alpha$ is continuous at $s = 0$.

Proof. We prove it by contradiction. Otherwise, we would have two sequences $\{s_n\}$ and $\{s_n'\}$ with $s_n, s_n' \to 0$ such that

$$\lim_{n\to\infty} I(s_n) \neq \lim_{n\to\infty} I(s_n').$$

By Proposition 1.13, we assume that $\varphi_{s_n} \to \varphi_0$, $\varphi_{s_n'} \to \varphi_0'$ in $C^{1,a}_{\mathrm{loc}}$ on $X_{0,\mathrm{reg}}$, then

$$\lim_{n\to\infty} I(s_n) = \int_{X_0} \varphi_0 \alpha, \quad \lim_{n\to\infty} I(s_n') = \int_{X_0} \varphi_0' \alpha.$$

Since $\varphi_0 - \varphi_0'$ is locally constant on $X_{0,\mathrm{reg}}$ and each irreducible component C of X_0 is connected, $\varphi_0 - \varphi_0'$ is piecewise constant on irreducible components of X_0. Therefore,

$$\int_{X_0} (\varphi_0 - \varphi_0')\alpha = \sum_C \text{constant} \int_C \alpha = 0,$$

where C runs over all irreducible components of X_0.

Thus, $\lim_{n\to\infty} I(s_n) - \lim_{n\to\infty} I(s_n') = 0$, which is a contradiction. □

1.D. **Smoothness of spectral projection**

Let $\pi\colon \mathfrak{X} \to S$ be a proper, smooth submersion between smooth manifolds. In Demailly's expository article [Dem02], he proved the following:

Lemma 1.15 ([Dem02, Lemma 10.3])**.** *Let g be a Riemannian metric on $\mathfrak{X}$, and let g_s be its restriction on each $X_s = \pi^{-1}(s)$. Then $-\Delta_s\colon C^\infty(X_s, \mathbb{R}) \to C^\infty(X_s, \mathbb{R})$ is a smooth family of elliptic operators. Furthermore, the eigenvalues of $-\Delta_s$*

$$0 < \lambda_0(s) \leq \lambda_1(s) \leq \cdots$$

are continuous in s.

Moreover, if λ is not in the spectrum $\{\lambda_k(s_0)\}_{k\geq 0}$ of $-\Delta_{s_0}$, the direct sum $W_{s,\lambda} \subset C^\infty(X_t)$ of eigenspaces of $-\Delta_s$ with eigenvalues $\leq \lambda$ defines a C^∞ vector bundle (i.e., varies smoothly), in a neighborhood of s_0.

As a direct corollary, we have

Corollary 1.16. *Let u be a smooth function on $\mathfrak{X}$ and $P^s_{\leq\lambda}\colon C^\infty(X_s) \to C^\infty(X_s)$ be the orthogonal projection onto $W_{s,\lambda}$.*

Suppose λ is not in the spectrum $\{\lambda_k(s_0)\}_{k\geq 0}$ of $-\Delta_{s_0}$, then $P^s_{\leq\lambda}(u)$ and $(1-P^s_{\leq\lambda})(u) =: P^s_{>\lambda}(u)$ are smooth on $\pi^{-1}(U)$, where $U \subset S$ is an open neighborhood of s_0.

2. Spectral geometry of degeneration of algebraic curves

In this section, we study the spectral geometry of the degeneration of algebraic curves, closely following the recent work of Dai–Yoshikawa [DY25]. In the first two subsections (Section 2.A, Section 2.B), we recall the tools developed in [DY25]. In the last subsection Section 2.C, we study the properties of eigenfunctions with small eigenvalues.

We introduce the setup for this section.

Setup 1. Let $\pi\colon X \to S \simeq \mathbb{D}$ be a proper surjective holomorphic map from a complex surface X to a Riemann surface isomorphic to the unit disk. We assume that π has connected fibers and $X_0 = \pi^{-1}(0)$ is the unique singular fiber. We define $\mathrm{Sing}(X_0) := \{x \in X_0 \mid d\pi_x = 0\}$, $X_{0,\mathrm{reg}} := X_0 \setminus \mathrm{Sing}(X_0)$ and $X^\circ := X \setminus X_0$. Let $X_s := \pi^{-1}(s)$.

Assume that X is equipped with a Kähler form ω, and let the corresponding metric be g^X. Let $g_s = g^X|_{X_s}$, then (X_s, g_s) $(s \neq 0)$ is a compact Riemann surface with a Kähler metric. We assume that $\mathrm{vol}(X_s, g_s) = \int_{X_s} \omega_s = 1$.

Assume that X_0 is a reduced and reducible divisor of X. In particular, π has only isolated critical points on X_0.

2.A. **Properties of small eigenvalues**

For $s \neq 0$, let $0 < \lambda_1(s) \leq \lambda_2(s) \leq \cdots$ be the eigenvalues of the Laplacian (with respect to g_s) $\square_s = \overline{\partial}^*\overline{\partial}$ counted with multiplicities. For $s = 0$, we regard $\square_0$ as the Friedrichs extension of the Laplacian on $X_{0,\mathrm{reg}} = X_0 \setminus \mathrm{Sing}\, X_0$ with compact support. Then by Yoshikawa's paper [Yos97], $\lambda_i(s)$ is a continuous function in $s \in S$ for each $i \geq 1$.

Let $N = \sharp\{\text{irreducible components of } X_0\} = \dim\ker(\square_0)$, we have a precise control for small eigenvalues around 0:

Theorem 2.1 ([DY25, Theorem 0.2])**.** *There exist constants $C_0, C_1 > 0$ such that for all $s \in \mathbb{D}^\circ$,*

$$\frac{C_0}{\log(|s|^{-1})} \leq \lambda_1(s) \leq \cdots \leq \lambda_{N-1}(s) \leq \frac{C_1}{\log(|s|^{-1})}$$

In contrast to the first $N-1$ small eigenvalues, the remaining eigenvalues have a lower bound as s approaches 0.

Theorem 2.2 ([DY25, Theorem 3.8])**.** *For all $k \geq 1$, the function $\lambda_k(s)$ on S° extends to a continuous function on S. Moreover, for all $k \geq N$, $\lambda_k(s) \geq \lambda$ for a positive number $\lambda > 0$.*

We call $\lambda_1(s), \cdots, \lambda_{N-1}(s)$ *small eigenvalues.* Using Corollary 1.16 and the spectral gap of eigenvalue $\lambda_N(s)$ in Theorem 2.2, we define the spectral projection onto the direct sum of eigenspaces with small eigenvalues.

Lemma 2.3. *We take a constant $0 < \delta_0 < 1$ depending only on the geometry of the degeneration and work on $\mathbb{D}^\circ_{\delta_0}$.*

For $s \in \mathbb{D}^\circ_{\delta_0}$, let $W_{s,\mathrm{low}} \subset C^\infty(X_s)$ be the space spanned by eigenfunctions with small eigenvalues $\lambda_1(s), \cdots, \lambda_{N-1}(s)$. Let $P^s_{\mathrm{low}} \colon C^\infty(X_s) \to W_{s,\mathrm{low}}$ be the orthogonal projection onto $W_{s,\mathrm{low}}$.

Then for any smooth function u on $X \setminus X_0$, its low frequency part $P^s_{\mathrm{low}}(u)$ is smooth on $\pi^{-1}(\mathbb{D}^\circ_{\delta_0})$. Similarly, its high frequency part $(1-P^s_{\mathrm{low}})(u)$ is also smooth on $\pi^{-1}(\mathbb{D}^\circ_{\delta_0})$.

Proof. (of Lemma 2.3) Using the notation in Theorem 2.1 and Theorem 2.2, we define δ_0 by

$$\mathbb{D}^\circ_{\delta_0} = \{s \in S^\circ \colon \frac{C_1}{\log(|s|^{-1})} \leq \frac{1}{2}\lambda\}$$

On the punctured disk $\mathbb{D}^\circ_{\delta_0}$, we have

$$\lambda_{N-1}(s) \leq \frac{C_1}{\log(|s|^{-1})} \leq \frac{1}{2}\lambda < \lambda_N(s).$$

For $s \in \mathbb{D}^\circ_{\delta_0}$, we have $P^s_{\mathrm{low}} = P^s_{\leq \frac{3}{4}\lambda}$. By Corollary 1.16, we know $P^s_{\leq \frac{3}{4}\lambda}(u)$ is smooth on $\pi^{-1}(\mathbb{D}^\circ_{\delta_0})$. □

Throughout this paper, we denote $P^s_{\mathrm{low}}(u)$ by u_{low} and $(1-P^s_{\mathrm{low}})(u)$ by u_{high}.

2.B. Thick-thin decomposition of nearby fibers

From a geometric perspective, each nearby smooth fiber X_s ($s \neq 0$) admits a thick-thin decomposition. As the family degenerates, the thick part of X_s converges to the regular locus of the singular fiber, $X_0 \setminus \mathrm{Sing}(X_0)$, while the thin part collapses to the singular points $\mathrm{Sing}(X_0)$. Since $X_0 \setminus \mathrm{Sing}(X_0)$ consists of N connected components, the thick part of X_s correspondingly decomposes into N disjoint regions.

Following [DY25, Section 6], for each $1 \leq i \leq N$, we construct a *model function* $\Upsilon^{(i)}_s \in C^\infty(X_s)$ in Proposition 2.8 that serves as a smooth approximation of the characteristic function on the i-th thick component.

We construct these *model functions* in the following three steps:

Step 1: We construct a family of diffeomorphisms which sends the smooth part of the singular fiber to nearby smooth fibers. The idea is to pick a C^∞ complex vector field v on $X\backslash\mathrm{Crit}(f)$ satisfying $f_*v = \partial/\partial s$, and use this vector field to flow points on the singular fiber to nearby smooth fibers.

We collect useful results in [DY25, Section 6] and omit the construction. Then we have the following theorem

Theorem 2.4. *For $s \in \mathbb{D}$ and $|s| \ll 1$. Let $\nu \in \mathbb{Z}_{\geq 1}$ be a positive integer depending on the degeneration. Let $\epsilon(s) = 2\,|s|^{\frac{1}{8\nu}}$, we have a family of diffeomorphisms*

$$F_s \colon X_0 \setminus \cup_{p\in \mathrm{Sing}\, X_0} B(p,\epsilon(s)) \to F_s(X_0 \setminus \cup_{p\in \mathrm{Sing}\, X_0} B(p,\epsilon(s)))(\subset X_s)$$
$$z \mapsto F_s(z) \in X_s$$

The family of diffeomorphisms enjoys the following properties:

(1) [DY25, Equation (6.2)] *When $s = 0$, then $F_0 \colon X_0 \to X_0$ is the identity map.*
(2) [DY25, Equation (6.2)] *The family F_s is smooth in s.*
(3) [DY25, Equation (6.3)] *For $z \in X_0$ in the domain of F_s, we have*

$$\mathrm{dist}(F_s(z), z) \leq K\,|s|^{\frac{3}{4}}$$

(4) [DY25, Lemma 6.4] *Let ω be the Kähler form in our settings, then*

$$\|(F_s)^*\omega_s - \omega_0\|_{L^\infty(X_0\setminus\cup_{p\in\mathrm{Sing}\,X_0}B(p,\epsilon(s)))} \leq K_2\,|s|^{\frac{1}{2}}\,.$$

(5) [DY25, Lemma 6.5] *For any two functions $\chi, \chi' \in C_0^\infty(X_0 \setminus \cup_{p\in\mathrm{Sing}\,X_0} B(p,\epsilon(s)))$, we have*

$$\left|((F_s)_*\chi, (F_s)_*\chi')_{L^2(X_s)} - (\chi,\chi')_{L^2(X_0)}\right| \leq K_3\,|s|^{\frac{1}{2}}\,\|\chi\|_{L^2(X_0)}\,\|\chi'\|_{L^2(X_0)}\,,$$
$$\left|\left\|\mathrm{d}\big((F_s)_*\chi\big)\right\|_{L^2(X_s)} - \|\mathrm{d}\chi\|_{L^2(X_0)}\right| \leq K_3\,|s|^{\frac{1}{2}}\,\|\mathrm{d}\chi\|_{L^2(X_0)}\,.$$

Remark 2.5. The result in Theorem 2.4.(4) can be generalized to any differential form on X with C^1 regularity. For any smooth form α on X with C^1-coefficients, we have

$$\|(F_s)^*\alpha_s - \alpha_0\|_{L^\infty(X_0\setminus\cup_{p\in\mathrm{Sing}\,X_0}B(p,\epsilon(s)))} \leq K_2\,|s|^{\frac{1}{2}}\,.$$

The proof follows verbatim as in [DY25, Lemma 6.4].

We estimate the volume of the subset in X_s that is not in the image of F_s.

Corollary 2.6. *Let O_s be the image of F_s inside X_s. Then O_s is open in X_s since F_s is a diffeomorphism. And for $|s| \ll 1$, there is a uniform constant $K_4 > 0$ such that*

$$\int_{X_s\setminus O_s} \omega_s \leq K_4\,|s|^{\frac{1}{4\nu}}\,.$$

Proof. Let $\chi_s \in C_0^\infty(X_0 \setminus \cup_{p\in\mathrm{Sing}\,X_0} B(p,\epsilon(s)))$ be a test function such that

- We have $0 \leq \chi_s \leq 1$.
- On $X_0 \setminus \cup_{p\in\mathrm{Sing}\,X_0} B(p,2\epsilon(s))$, $\chi_s \equiv 1$.

Then we have

$$0 \leq \mathrm{vol}(X_0) - (\chi_s,\chi_s)_{L^2(X_0)} \leq \sum_{p\in\mathrm{Sing}\,X_0} K_4'(\epsilon(s))^2 = K_4'\,|s|^{\frac{1}{4\nu}}\,,$$

where K_4' is a constant depending on the geometry of singular fiber.

Since X_0 is reduced and ω is Kähler, we have $\mathrm{vol}(X_0) = \mathrm{vol}(X_s)$. Together with Theorem 2.4.(5), we have

$$0 \le \mathrm{vol}(X_s) - ((F_s)_*\chi_s, (F_s)_*\chi_s)_{L^2(X_s)} \le K_3 \left|s\right|^{\frac{1}{2}} + K_4' \left|s\right|^{\frac{1}{4\nu}} .$$

Note that $0 \le (F_s)_*\chi \le 1$ and $(F_s)_*\chi \in C_0^\infty(O_s)$, we have

$$\int_{X_s \setminus O_s} \omega_s \le \mathrm{vol}(X_s) - ((F_s)_*\chi, (F_s)_*\chi)_{L^2(X_s)} \le K_4 \left|s\right|^{\frac{1}{4\nu}} .$$

□

Step 2: We construct good test functions on the singular fiber, which will be flowed using diffeomorphisms in Theorem 2.4 to nearby fibers in **Step 3** for constructing *model functions*.

Let $X_0 = C_1 + \cdots + C_N$ be the irreducible decomposition of the singular fiber X_0. For each singular point $p \in \mathrm{Sing}\, X_0 \cap C_i$, we fix a local coordinate system $(U_p, \zeta), \zeta = (\zeta_1, \zeta_2) \in \mathbb{C}^2$ and $\zeta(p) = 0$. We define $r_p(z) := \|\zeta(z)\| = \sqrt{\left|\zeta_1\right|^2 + \left|\zeta_2\right|^2}$.

Then we construct a family of functions on X_0.

Lemma 2.7 ([DY25, Lemma 6.6])**.** *For every $0 < \epsilon \ll 1$, there exists a family of smooth functions $\chi_\epsilon^{(i)} (1 \le i \le N)$, such that*

(1) For $1 \le i \le N$, we have $\chi_\epsilon^{(i)} \in C_0^\infty(C_i \setminus \mathrm{Sing}(X_0))$
(2) We have $0 \le \chi_\epsilon^{(i)} \le 1$. On $C_i \setminus \bigcup_{p \in \mathrm{Sing}\, X_0 \cap C_i} U_p$, we have $\chi_\epsilon^{(i)} = 1$.
(3) For any $p \in \mathrm{Sing}\, X_0 \cap C_i$, we have $\chi_\epsilon^{(i)}(z) = 0$ if $r_p(z) \le \frac{1}{2}\epsilon$ and $\chi_\epsilon^{(i)}(z) = 1$ if $r_p(z) \ge 2\sqrt{\epsilon}$.
(4) We have $\left\|\mathrm{d}\chi_\epsilon^{(i)}\right\|_{L^2}^2 \le K_4/(\log \epsilon^{-1})$ where $K_4 > 0$ is a constant depending only on $\pi\colon X \to S$ around $0 \in S$.
(5) For each i, the $\chi_\epsilon^{(i)}$ depends on ϵ continuously.

Step 3: We use the family of diffeomorphisms generated by flow in **Step 1** and the collection of good test functions on X_0 in **Step 2** to construct the collection of *model functions* on X_s.

Proposition 2.8 ([DY25, Section 6.2])**.** *For $0 < |s| \ll 1$, there exists a family of smooth functions $\Upsilon_s^{(i)} \in C^\infty(X_s)$ $(1 \le i \le N)$, such that*

(1) For any $1 \le i \le N$ and any s, there is a constant $C > 0$ depending only on the geometry of degeneration such that $0 \le \Upsilon_s^{(i)} \le C$.
(2) For $1 \le i \le N$, the function $\Upsilon^{(i)}(s, z) := \Upsilon_s^{(i)}(z)$ for $0 < |s| \ll 1, z \in X_s$ is continuous on $X \setminus X_0$, i.e., the family varies continuously in s away from the singular fiber.
(3) For $1 \le i, j \le N$, $i \ne j$, we have $\mathrm{Supp}\, \Upsilon_s^{(i)} \cap \mathrm{Supp}\, \Upsilon_s^{(j)} = \emptyset$.

(4) For $1 \leq i \leq N$*, we have estimates*

$$\left\|\Upsilon_s^{(i)}\right\|_{L^2(X_s)}^2 = 1 + O(|s|^{\frac{1}{8\nu}}), \quad \left\|\mathrm{d}\Upsilon_s^{(i)}\right\|_{L^2(X_s)}^2 \leq \frac{K}{\log(|s|^{-1})}.$$

The constant $K > 0$ *depends only on* $(\pi \colon X \to \mathbb{D}, \omega)$*.*

(5) The function $\Upsilon_s^{(i)}(z)$ *is constant for* $z \in X_s$ *in* $F_s(X_0 \setminus \bigcup_{p \in \operatorname{Sing} X_0} B(p, 4\epsilon(s)^{\frac{1}{2}}))$*, where* $\epsilon(s) = 2\,|s|^{\frac{1}{8\nu}}$*.*

Proof. We define

$$\Upsilon^{(i)}(s) = (F_s)_*(\chi_{4\epsilon(s)}^{(i)})/\sqrt{\operatorname{Area}(C_i)} \in C^\infty(X_s), \tag{2.1}$$

where $\epsilon(s) = 2\,|s|^{\frac{1}{8\nu}}$.

Since $0 \leq \chi_{\epsilon(s)}^{(i)} \leq 1$, Proposition 2.8.(1) is proved.

Since $\chi_{\epsilon(s)}^{(i)}$ depends on s continuously by Lemma 2.7.(5) and F_s is continuous in s, $\Upsilon^{(i)}(s)$ varies continuously in s. Then Proposition 2.8.(2) is proved.

For the remaining parts, we refer to [DY25, Section 6.2]. □

Corollary 2.9. *For any* $p \in X_0 \setminus \operatorname{Sing} X_0$*, there is an open neighborhood* U *of* p *such that* $\Upsilon^{(i)}(s,z) := \Upsilon_s^{(i)}(z)$ *is constant for* $1 \leq i \leq N$*.*

Moreover, we can extend $\Upsilon^{(i)}(s,z)$ *to be a continuous function on* $X \setminus \operatorname{Sing} X_0$*. And* $\Upsilon^{(i)}(s,z)$ *is locally constant around* $X_0 \setminus \operatorname{Sing} X_0$*.*

Proof. By the definition of F_s in Theorem 2.4, for any p in $X_0 \setminus \operatorname{Sing} X_0$, we can take an open neighborhood U of p in X such that $U \cap X_s$ is contained in $F_s(X_0 \setminus \bigcup_{p \in \operatorname{Sing} X_0} B(p, 4\epsilon(s)^{\frac{1}{2}}))$. From the definition of $\Upsilon^{(i)}$ in Equation (2.1), the $\Upsilon^{(i)}$ equals to $1/\sqrt{\operatorname{Area}(C_i)}$ on U. So $\Upsilon^{(i)}$ is locally constant around any point in $X \setminus \operatorname{Sing} X_0$.

Together with Proposition 2.8.(2), we know that $\Upsilon^{(i)}$ is continuous on $X \setminus \operatorname{Sing} X_0$. Furthermore, it equals to $1/\sqrt{\operatorname{Area}(C_i)}$ around any point in $X \setminus \operatorname{Sing} X_0$. □

We call $\Upsilon_s^{(i)}$ *model functions.*

2.C. Properties of small eigenfunctions

Eigenfunctions associated with small eigenvalues are called *small eigenfunctions*. We approximate these using the *model functions* defined in Proposition 2.8. The results are summarized below.

Theorem 2.10. *Let* $\{\Phi_i(s,z)\}_{i=1}^{N-1}$ *be eigenfunctions of* Δ_{ω_s} *with eigenvalues* $0 < \lambda_1(s) \leq \lambda_2(s) \leq \cdots \leq \lambda_{N-1}(s)$*, i.e.,* $-\Delta_{\omega_s}\Phi_i = \lambda_i(s)\Phi_i$*, on* X_s*. Then for each* $1 \leq j \leq N-1$*, we have a decomposition for* Φ_j*,*

$$\Phi_j(s,z) = \sum_{l=1}^{N} C_{l(j+1)}(s)\Upsilon^{(l)}(s,z) + R_j(s,z) \tag{2.2}$$

for $0 < |s| \ll 1$*, where* $C_{ij}(s) = \int_{X_s} \Upsilon_s^{(i)}\Phi_{j-1}\omega_s = (\Upsilon_s^{(i)}, \Phi_{j-1})_{L^2(X_s,\omega_s)}$*. And we have the following estimates*

(1) There is a uniform constant C *such that* $|C_{lj}(s)| < C$ *for all* $1 \leq l \leq N$ *and all* s*.*

(2) There is a uniform constant C such that

$$\|\mathrm{d}R_j\|^2_{L^2(X_s)} \leq \frac{C^2}{\log(|s|^{-1})}, \quad \|R_j\|^2_{L^2(X_s)} \leq C^2 \frac{1}{\log(|s|^{-1})}.$$

for all $1 \leq j \leq N$ and all s.

Remark 2.11. We mention that the choice of small eigenfunctions is not unique. Nevertheless, the non-uniqueness of choice can be controlled: Let $\{\Phi_i(s,z)\}_{i=1}^{N-1}$ and $\{\Phi'_i(s,z)\}_{i=1}^{N-1}$ be two families of normalized small eigenfunctions. They span the same vector space V and are orthonormal bases under the L^2-inner product. Thus, they are related by an orthogonal matrix, i.e., there exists a matrix $O \in O(N-1)$ such that

$$(\Phi'_1(s,z), \Phi'_2(s,z), \cdots, \Phi'_{N-1}(s,z)) = (\Phi_1(s,z), \Phi_2(s,z), \cdots, \Phi_{N-1}(s,z))O.$$

From the equation Equation (2.2), we have

$$\begin{aligned}(\Phi'_1, \Phi'_2, \cdots, \Phi'_{N-1}) &= (\Upsilon^{(1)}, \cdots, \Upsilon^{(N)})(C_{l(j+1)})_{\substack{1\leq l\leq N\\ 1\leq j\leq N-1}} O + (R_1, \cdots, R_{N-1})O \\ &= (\Upsilon^{(1)}, \cdots, \Upsilon^{(N)})(C'_{l(j+1)})_{\substack{1\leq l\leq N\\ 1\leq j\leq N-1}} + (R'_1, \cdots, R'_{N-1}).\end{aligned}$$

So for another choice of eigenfunctions, the remainders are related by an orthogonal matrix and the estimates are the same for $R'_1, \cdots, R'_{N-1}$.

Proof. We begin with the Fourier expansion for each model function $\Upsilon_s^{(i)}$ $(1 \leq i \leq N)$ on X_s with respect to the lower frequency eigenfunctions $\{\Phi_j(s,z)\}_{j=1}^{N-1}$,

$$\Upsilon_s^{(i)} = \underbrace{\int_{X_s} \Upsilon_s^{(i)} \omega_s + \sum_{j=1}^{N-1} c_{ij}(s)\Phi_j}_{=:\Upsilon^{(i)}_{s,\mathrm{low}}} + h_s^{(i)}, \tag{2.3}$$

where $c_{ij}(s) = \int_{X_s} \Upsilon_s^{(i)} \Phi_j \omega_s$ and $h_s^{(i)}$ is the high frequency remainders.

We estimate $h_s^{(i)}$ in the following lemma:

Lemma 2.12. *There is a uniform constant $K > 0$ depending only on the degeneration such that*

$$\|h_s^{(i)}\|^2_{L^2(X_s,\omega_s)} \leq \frac{K}{\log(|s|^{-1})}$$

Proof. (of Lemma 2.12) Applying $-\Delta_{\omega_s}$ to Equation (2.3), we get the following equation on X_s

$$-\Delta_{\omega_s} \Upsilon_s^{(i)} = \sum_{j=1}^{N-1} c_{ij}(s)\lambda_j(s)\Phi_j - \Delta_{\omega_s} h_s^{(i)}.$$

Since $h_s^{(i)}$ has no low frequency modes, by spectral gap in Theorem 2.2, we have a uniform $\lambda > 0$ such that

$$\begin{aligned} \lambda \left\|h_s^{(i)}\right\|^2_{L^2(X_s,\omega_s)} &\le (-\Delta_{\omega_s} h_s^{(i)}, h_s^{(i)})_{L^2(X_s)} \\ &= -\left(\Delta_{\omega_s}\Upsilon_s^{(i)} + \sum_{j=1}^{N-1} c_{ij}(s)\lambda_j(s)\Phi_j, h_s^{(i)}\right)_{L^2(X_s)} \\ &= \left(-\Delta_{\omega_s}\Upsilon_s^{(i)}, h_s^{(i)}\right)_{L^2(X_s)} \\ &= \left(-\Delta_{\omega_s}\Upsilon_s^{(i)}, \Upsilon_s^{(i)}\right)_{L^2(X_s)} + \left(\Delta_{\omega_s}\Upsilon_s^{(i)}, \Upsilon_{s,\mathrm{low}}^{(i)}\right)_{L^2(X_s)} \end{aligned} \tag{2.4}$$

The equality in the third line holds since the high frequency term $h_s^{(i)}$ in the Fourier expansion (Equation (2.3)) is $L^2(X_s,\omega_s)$-orthogonal to low frequency modes Φ_j ($1 \le j \le N-1$).

Integrating by parts on the compact manifold X_s and using the estimate for the model functions $\Upsilon_s^{(i)}$ (in Proposition 2.8(4)), we get

$$\left(-\Delta_{\omega_s}\Upsilon_s^{(i)}, \Upsilon_s^{(i)}\right)_{L^2(X_s)} = \left\|\mathrm{d}\Upsilon_s^{(i)}\right\|^2_{L^2(X_s)} \le \frac{K}{\log(|s|^{-1})},$$

where K is a uniform constant.

Since Δ_{ω_s} preserves orthogonality in the Fourier expansion Equation (2.3), we have

$$\begin{aligned} \left(\Delta_{\omega_s}\Upsilon_s^{(i)}, \Upsilon_{s,\mathrm{low}}^{(i)}\right)_{L^2(X_s)} &= \left(\Delta_{\omega_s}\Upsilon_{s,\mathrm{low}}^{(i)}, \Upsilon_{s,\mathrm{low}}^{(i)}\right)_{L^2(X_s)} \\ &= -\left\|\mathrm{d}\Upsilon_{s,\mathrm{low}}^{(i)}\right\|^2_{L^2(X_s)} \le 0. \end{aligned}$$

Put things back into Equation (2.4), we get

$$\lambda \left\|h_s^{(i)}\right\|^2_{L^2(X_s,\omega_s)} \le \frac{K}{\log(|s|^{-1})}.$$

So we can find a uniform $K_1 > 0$ such that for all $1 \le i \le N$, we have

$$\left\|h_s^{(i)}\right\|^2_{L^2(X_s,\omega_s)} \le \frac{K'}{\log(|s|^{-1})}.$$

□

We now go back to our theorem.

(1) We use the notation $\Phi_0 \coloneqq 1$ and consider the following correlation matrix $C(s) = (C_{ij}(s))_{1\le i,j\le N}$,

$$C_{ij}(s) = \int_{X_s} \Upsilon_s^{(i)}\Phi_{j-1}\omega_s = (\Upsilon_s^{(i)}, \Phi_{j-1})_{L^2(X_s,\omega_s)}.$$

Then $|C_{ij}(s)| \le \left\|\Upsilon_s^{(i)}\right\|_{L^2} \|\Phi_{j-1}\|_{L^2} = O(1)$.

(2) We then show that the correlation matrix $C(s)$ is close to an orthogonal matrix with controlled error terms:

$$
\begin{aligned}
(CC^T)_{ij} &= \sum_{l=1}^{N} C_{il}C_{jl} \\
&= \sum_{l=1}^{N-1} (\Upsilon_s^{(i)}, \Phi_{l-1})_{L^2(X_s)} (\Upsilon_s^{(j)}, \Phi_{l-1})_{L^2(X_s)} \\
&= (\Upsilon_s^{(i)}, \sum_{l=1}^{N} \Phi_{l-1}(\Upsilon_s^{(j)}, \Phi_{l-1})) \\
&= (\Upsilon_s^{(i)}, \Upsilon_s^{(j)} - h_s^{(j)}) \\
&= \|\Upsilon_s^{(i)}\|^2_{L^2(X_s)} \delta_{ij} - (\Upsilon_s^{(i)}, h_s^{(j)})
\end{aligned}
$$

The last equality follows by Proposition 2.8(3).

Since $\left\|\Upsilon_s^{(i)}\right\|^2_{L^2(X_s)} = 1 + O(|s|^{\frac{1}{8\nu}})$ (by Proposition 2.8(4)) and $\left|(\Upsilon_s^{(i)}, h_s^{(j)})\right| \leq \left\|h_s^{(j)}\right\|_{L^2} \left\|\Upsilon_s^{(i)}\right\|_{L^2} = O(\frac{1}{\log(s^{-1})})^{\frac{1}{2}}$ (by Lemma 2.12), we have

$$CC^T = I + E,$$

where E, the error term, is controlled by $\|E\| \leq K_1(\frac{1}{\log(s^{-1})})^{\frac{1}{2}}$. So $C^{-1} = C^T(I+E)^{-1}$.

Since $\|E\| \ll 1$ for $|s| \ll 1$, we have

$$(I+E)^{-1} = I - E + E^2 - E^3 + \cdots =: I + E',$$

where $E' = \sum_{i\geq 1}(-1)^i E^i$. We estimate that

$$\|E'\| \leq \|E\| \, (1 + \sum_{i\geq 1} \|E\|^i) \leq K_2(\frac{1}{\log(s^{-1})})^{\frac{1}{2}}$$

for some uniform constant K_2. Then

$$C^{-1} = C^T(I + E').$$

Note that

$$
\begin{pmatrix} \Upsilon_s^{(1)} - h_s^{(1)} \\ \Upsilon_s^{(2)} - h_s^{(2)} \\ \vdots \\ \Upsilon_s^{(N)} - h_s^{(N)} \end{pmatrix}
= \begin{pmatrix} \Upsilon_{s,\text{low}}^{(1)} \\ \Upsilon_{s,\text{low}}^{(2)} \\ \vdots \\ \Upsilon_{s,\text{low}}^{(N)} \end{pmatrix}
= C(s) \begin{pmatrix} \Phi_0 \\ \Phi_1 \\ \vdots \\ \Phi_{N-1} \end{pmatrix}
= C(s) \begin{pmatrix} 1 \\ \Phi_1 \\ \vdots \\ \Phi_{N-1} \end{pmatrix}
$$

We have

$$
\begin{pmatrix} 1 \\ \Phi_1 \\ \vdots \\ \Phi_{N-1} \end{pmatrix}
= C^{-1}(s) \begin{pmatrix} \Upsilon_{s,\text{low}}^{(1)} \\ \Upsilon_{s,\text{low}}^{(2)} \\ \vdots \\ \Upsilon_{s,\text{low}}^{(N)} \end{pmatrix}
= C^T(s)(I + E') \begin{pmatrix} \Upsilon_{s,\text{low}}^{(1)} \\ \Upsilon_{s,\text{low}}^{(2)} \\ \vdots \\ \Upsilon_{s,\text{low}}^{(N)} \end{pmatrix}
$$

So for each $1 \leq j \leq N-1$, we have

$$\Phi_j(s) = \sum_{l=1}^{N} C_{l(j+1)}(s)\Upsilon^{(l)}_{s,\mathrm{low}} + R_j. \tag{2.5}$$

And the remainder term $R_j = \left(C^T \cdot E' \cdot (\Upsilon^{(1)}_{s,\mathrm{low}}, \cdots, \Upsilon^{(N)}_{s,\mathrm{low}})^T\right)_{j+1}$ is controlled by

$$|R_j| \leq \left\|C^T\right\|_{L^\infty} \|E'\|_{L^\infty} \left\|(\Upsilon^{(1)}_{s,\mathrm{low}}, \cdots, \Upsilon^{(N)}_{s,\mathrm{low}})^T\right\|_{L^\infty} = O(\frac{1}{\log(s^{-1})})^{\frac{1}{2}}.$$

So $\|R_j\|_{L^2(X_s)} \leq C^2 \frac{1}{\log(|s|^{-1})}$ for some constant $C > 0$.

To estimate the derivative of the remainder, we take the derivative of the Equation (2.5) on X_s,

$$\mathrm{d}\Phi_j(s) - C_{l(j+1)}(s) \sum_{l=1}^{N} \mathrm{d}\Upsilon^{(l)}_{s,\mathrm{low}} = \mathrm{d}R_j.$$

Then

$$\|\mathrm{d}R_j\|_{L^2} \leq \sum_{l=1}^{N} \left\|C_{l(j+1)}\right\|_{L^\infty} \left\|\mathrm{d}\Upsilon^{(l)}_{s,\mathrm{low}}\right\|_{L^2} + \|\mathrm{d}\Phi_j(s)\|_{L^2}.$$

Since $\|\mathrm{d}\Phi_j(s)\|_{L^2} = \lambda_j^{\frac{1}{2}}(s)$ and $\left\|\mathrm{d}\Upsilon^{(l)}_{s,\mathrm{low}}\right\|^2_{L^2} = O\left(\frac{1}{\log(|s|^{-1})}\right)$ by Proposition 2.8(4), we have

$$\|\mathrm{d}R_j\|^2_{L^2} \leq C^2 \frac{1}{\log(|s|^{-1})}$$

for some constant $C > 0$.

□

3. Estimates of preferred potentials

The setup for this section is the same as Section 2 (see Setup 1).

Given a smooth real $(1,1)$-form α on X, we introduce the following two integrability conditions:

Condition 3.1. We say α satisfies *the integrability condition on general fibers*, if $\int_{X_s} \alpha = 0$ for all $s \neq 0$.

Condition 3.2. We say α satisfies *the integrability condition on the singular fiber*, if α satisfies *the integrability condition on general fibers (Condition 3.1)* and

$$\int_{C_i} \alpha = 0,$$

where C_i $(1 \leq i \leq N)$ are irreducible components of the reduced singular fiber X_0.

Our goal in this section is to study the *preferred potentials* of smooth $(1,1)$-forms on X satisfying Condition 3.1. The notion of *preferred potentials* originates from [FT23, Theorem 3.2.4].

Definition 3.3 (**Preferred potentials**). *Let α be a smooth real $(1,1)$-form on (X, ω) satisfying* the integrability condition on general fibers (Condition 3.1). *Then there is* *a unique* $\varphi \in C^\infty(X \setminus X_0)$ *such that*

$$\int_{X_s} \varphi\omega_s = 0, \quad (\alpha + \mathrm{dd}^c\varphi) \mid_{X_s} = 0.$$

We say that φ *is the* preferred potential *of the differential form* α.

Since (X_s, ω_s) is compact and $\varphi \mid_{X_s} =: \varphi_s$ is smooth on it, we can expand it using Fourier series in terms of eigenfunctions of Δ_{ω_s}:

$$\varphi_s = \sum_{i=1}^{N-1} c_i(s)\Phi_i(s,z) + \sum_{i \geq N} c_i(s)\Phi_i(s,z).$$

The coefficients are determined by

$$\begin{aligned} c_i(s) &= \int_{X_s} \varphi_s \Phi_i(s,z)\omega_s \\ &= 2\pi \int_{X_s} \varphi_s \frac{-\mathrm{dd}^c_z \Phi_i(s,z)}{\lambda_i(s)} \\ &= 2\pi \int_{X_s} \mathrm{dd}^c_z(\varphi_s) \frac{-\Phi_i(s,z)}{\lambda_i(s)} \\ &= \frac{2\pi}{\lambda_i(s)} \int_{X_s} \Phi_i(s,z)\alpha. \end{aligned}$$

We use $b_i(s) \coloneqq \int_{X_s} \Phi_i(s,z)\alpha$ to denote the integral terms, and the properties of $\lambda_i(s)$ are known for $1 \leq i \leq N-1$ in Theorem 2.1.

We then divide φ into two parts, a low frequency part and a high frequency part, following the Fourier expansion:

$$\varphi_{\text{low}} = \sum_{i=1}^{N-1} c_i(s)\Phi_i(s,z), \quad \varphi_{\text{high}} = \sum_{i \geq N} c_i(s)\Phi_i(s,z). \tag{3.1}$$

These two parts are smooth on $X \setminus X_0$ by Lemma 2.3.

Then we introduce the main theorems of this section.

Theorem 3.4. *We work under Setup 1. Suppose* α *is a smooth real* $(1,1)$*-form on* (X, ω) *with Condition 3.1. Let* φ *be the preferred potential of* α.

(1) We have an L^∞ *bound for* φ_{high},

$$\|\varphi_{\text{high}}\|_{L^\infty(X_s)} \leq C \|\mathrm{tr}_{\omega_s}\alpha\|_{L^\infty(X_s)}$$

for all $s \neq 0$. *The constant* C *only depends on the geometry of the degeneration.*

(2) We have control over the growth order of $\|\varphi_{\text{low}}\|_{L^\infty(X_s)}$,

$$\|\varphi_{\text{low}}\|_{L^\infty(X_s)} \leq C \|\mathrm{tr}_{\omega_0}\alpha\|_{L^1(X_0)} \log(|s|^{-1}).$$

for all $s \neq 0$. *The constant* C *only depends on the geometry of the degeneration.*

When α satisfies the integrability condition on the singular fiber (Condition 3.2), the growth order of $\|\varphi_{\text{low}}\|_{L^\infty(X_s)}$ has a better control.

Theorem 3.5. *We use the same assumption and notations in Theorem 3.4. We assume that the smooth real* $(1,1)$*-form* α *satisfies Condition 3.2. Then the low frequency* *part φ_{low} of the preferred potential satisfies*

$$\frac{\|\varphi_{\mathrm{low}}\|_{L^\infty(X_s)}}{\log^{1/2}(|s|^{-1})} \to 0, \quad \textit{as } s \to 0.$$

3.A. **Estimate of the high frequency part**

We prove the first part of Theorem 3.4 regarding the high frequency part of the *preferred potential*. The proof is based on Cheng-Li's argument [CL81] (also seen in Siu's lecture note [Siu87, Appendix (A.2)]).

First, we introduce the following high frequency truncated heat kernel and Green's function on each (X_s, ω_s):

$$H_s^N(z_1, z_2, t) = \sum_{i=N}^{\infty} e^{-\lambda_i(s)t}\Phi_i(s, z_1)\Phi_i(s, z_2) \quad \text{(truncated heat kernel)}$$

$$G_s^N(z_1, z_2) = \sum_{i=N}^{\infty} \frac{1}{\lambda_n(s)}\Phi_i(s, z_1)\Phi_i(s, z_2) \quad \text{(truncated Green's function)}$$

Then the high frequency part φ_{high} in Equation (3.1) can be represented by

$$\varphi_{\mathrm{high}}(z, s) = 2\pi \int_{z_2 \in X_s} (\mathrm{tr}_{\omega_s}\alpha)(z_2) G_s^N(z, z_2)\omega_s(z_2) \tag{3.2}$$

To obtain the desired bound on φ_{high} in Theorem 3.4, we estimate $G_s^N(z_1, z_2)$ using Cheng-Li's argument:

Proposition 3.6. *There is a uniform constant $C > 0$ independent of $s \in \mathbb{D}^\circ$ such that*

$$G_s^N(z_1, z_2) \geq -C, \quad z_1, z_2 \in X_s,$$

for all $s \in \mathbb{D}^\circ$.

Proof. Let $H_s(z_1, z_2, t)$ be the heat kernel of (X_s, ω_s), then we have

$$\frac{1}{V(s)} + \sum_{i=1}^{N-1} e^{-\lambda_i(s)t}\Phi_i(s, z_1)\Phi_i(s, z_2) + H_s^N(z_1, z_2, t) = H_s(z_1, z_2, t) \geq 0, \tag{3.3}$$

where $V(s) = \int_{X_s} \omega_s$ is the volume of X_s. Since ω is closed, $V(s) \equiv V$ is constant.

From the lower bound of higher eigenvalues in Theorem 2.2, we know that there exists $\lambda > 0$ such that $\lambda_k(s) \geq \lambda$ for all $k \geq N$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t} H_s^N(z, z, t) = \sum_{i=N}^{\infty} -\lambda_i(s)e^{-\lambda_i(s)t}\left|\Phi_i(s, z)\right|^2 \leq -\lambda H_s^N(z, z, t).$$

By Grönwall's inequality, we have

$$H_s^N(z, z, t) \leq H_s^N(z, z, 1)e^{-\lambda(t-1)} \quad \text{for } t \geq 1$$

By the Cauchy-Schwarz inequality, we get

$$\begin{aligned} \left|H_s^N(z_1, z_2, t)\right| &\leq \sqrt{H_s^N(z_1, z_1, t) \cdot H_s^N(z_2, z_2, t)} \\ &\leq \sqrt{H_s^N(z_1, z_1, 1) \cdot H_s^N(z_2, z_2, 1)}e^{-\lambda(t-1)} \quad \text{for } t \geq 1. \end{aligned} \tag{3.4}$$

To estimate the truncated Green's function, we note that

$$\int_0^\infty H_s^N(z_1,z_2,t)\mathrm{d}t = G_s^N(z_1,z_2) = \sum_{i=N}^\infty \frac{1}{\lambda_i(s)}\Phi_i(s,z_1)\Phi_i(s,z_2).$$

Then

$$\begin{aligned}
G_s^N(z_1,z_2) &= \int_0^1 H_s^N(z_1,z_2,t)\mathrm{d}t + \int_1^\infty H_s^N(z_1,z_2,t)\mathrm{d}t \\
&\geq \int_0^1 H_s^N(z_1,z_2,t)\mathrm{d}t - \sqrt{H_s^N(z_1,z_1,1)\cdot H_s^N(z_2,z_2,1)}\int_1^\infty e^{-\lambda(t-1)}\mathrm{d}t \\
&= \int_0^1 \big(H_s(z_1,z_2,t) - \frac{1}{V} - \sum_{i=1}^{N-1} e^{-\lambda_i(s)t}\Phi_i(s,z_1)\Phi_i(s,z_2)\big)\mathrm{d}t - I_2 \\
&\quad (\text{Using } \textit{Equation } (3.3), \text{and } I_2 := \sqrt{H_s^N(z_1,z_1,1)\cdot H_s^N(z_2,z_2,1)}\int_1^\infty e^{-\lambda(t-1)}\mathrm{d}t) \\
&\geq \int_0^1 \big(-\frac{1}{V} - \sum_{i=1}^{N-1} e^{-\lambda_i(s)t}\Phi_i(s,z_1)\Phi_i(s,z_2)\big)\mathrm{d}t - I_2 \\
&\quad (\text{Using the fact that the heat kernel } H_s(z_1,z_2,t)\geq 0) \\
&\geq -\frac{1}{V} - \sum_{i=1}^{N-1} \|\Phi_i(s,z)\|^2_{L^\infty(X_s)} - I_2 \\
&\geq -(\frac{1}{V} + I_1 + I_2)
\end{aligned}$$

Here

$$\begin{aligned}
I_1 &= \sum_{i=1}^{N-1} \|\Phi_i(s,z)\|^2_{L^\infty(X_s)} \geq 0 \\
I_2 &= \sqrt{H_s^N(z_1,z_1,1)\cdot H_s^N(z_2,z_2,1)}\int_1^\infty e^{-\lambda(t-1)}\mathrm{d}t \geq 0.
\end{aligned}$$

It remains to control I_1, I_2 by some uniform constants that are independent of s.

We begin with basic results which relate the Sobolev constants to the heat kernel and eigenfunction bounds.

Theorem 3.7 (Theorem 2.1, [Yos97])**.** *For a manifold (M,g) of real dimension 2, the following two inequalities are equivalent:*

(1) For $0<t\leq 1$ and $(x,y)\in M\times M$,

$$H(x,y,t)\leq C_1 t^{-2}$$

(2) For every $f\in C^\infty(M)$,

$$\|f\|_{L^4}\leq C_2(\|\mathrm{d}f\|_{L^2}+\|f\|_{L^2})$$

Here C_1 and C_2 depend continuously on each other.

By classical Moser iteration, we have

Theorem 3.8 ([Pet16, Theorem 9.2.7]). *Let (M, g) be a compact Riemannian manifold, with the following Sobolev inequality*

$$\|u\|_{2\nu} \le S\|\nabla u\|_2 + \|u\|_2$$

where $\nu > 1$. Then for $\Delta u = -\lambda u$, we have

$$\|u\|_\infty \le \exp\left(\frac{S\sqrt{\nu\lambda}}{\sqrt{\nu}-1}\right)\|u\|_2 .$$

Specifically, if $\nu = 2$ in the above Sobolev inequality, and $\|u\|_2 = 1$, then for $\Delta u = -\lambda u$, we have

$$\|u\|_\infty \le \exp\left(\frac{S\sqrt{2\lambda}}{\sqrt{2}-1}\right)$$

Finally, we introduce the uniform Sobolev inequality for a Kähler family.

Theorem 3.9 ([Tos10, Lemma 3.2]). *Let $\pi : (X, \omega) \to \mathbb{D}$ be a Kähler family, i.e., π is a proper, surjective holomorphic map from a Kähler manifold (X, ω) to the unit disk $\mathbb{D}$. Suppose that $X_0 := \pi^{-1}(0)$ is the unique singular fiber.*

Assume $\dim_{\mathbb{C}} X_s = 1$. *Then for $0 < |s| < 0.5$, we have a uniform Sobolev constant C for (X_s, ω_s), where $\omega_s = \omega|_{X_s}$:*

$$\|f\|_{L^4(X_s)} \le C(\|\mathrm{d}f\|_{L^2} + \|f\|_{L^2}) \quad \forall f \in C^\infty(X_s)$$

So I_1 is bounded uniformly by Theorem 3.9 and Theorem 3.8. And I_2 is bounded uniformly by Theorem 3.9 and Theorem 3.7. This completes the proof. □

As a corollary, we can show the first part of Theorem 3.4.

Corollary 3.10. *Under the settings and notation of Theorem 3.4, we have*

$$\|\varphi_{\text{high}}\|_{L^\infty(X_s)} \le C\,\|\mathrm{tr}_{\omega_s}\alpha\|_{L^\infty(X_s)}$$

for all $s \neq 0$. The constant C only depends on the geometry of the degeneration.

Proof. By Equation (3.2), we have

$$\begin{aligned}\varphi_{\text{high}}(z, s) &= 2\pi \int_{z_2 \in X_s} (\mathrm{tr}_{\omega_s}\alpha)(z_2) G_s^N(z, z_2)\omega_s(z_2) \\ &= 2\pi \int_{z_2 \in X_s} (\mathrm{tr}_{\omega_s}\alpha)(z_2)(G_s^N(z, z_2) + C')\omega_s(z_2),\end{aligned}$$

where we take the constant C' as in Proposition 3.6 such that $G_s^N(z, z_2) + C' \ge 0$.

So

$$\begin{aligned}\varphi_{\text{high}}(z, s) &\le 2\pi \int_{z_2 \in X_s} |(\mathrm{tr}_{\omega_s}\alpha)(z_2)|\,(G_s^N(z, z_2) + C')\omega_s(z_2) \\ &\le 2\pi \int_{z_2 \in X_s} \|\mathrm{tr}_{\omega_s}\alpha\|_{L^\infty(X_s)}\,(G_s^N(z, z_2) + C')\omega_s(z_2) \\ &= C'V\,\|\mathrm{tr}_{\omega_s}\alpha\|_{L^\infty(X_s)} .\end{aligned}$$

Similarly, we have $\varphi_{\text{high}}(z, s) \ge -C'V\,\|\mathrm{tr}_{\omega_s}\alpha\|_{L^\infty(X_s)}$, where V is the volume of (X_s, ω_s).

To summarize, we have $\|\varphi_{\text{high}}\|_{L^\infty(X_s)} \le C\,\|\mathrm{tr}_{\omega_s}\alpha\|_{L^\infty(X_s)}$. □

3.B. Estimate of the low frequency part (I)

We prove the second part of Theorem 3.4 here.

First, we note that

$$\varphi_{\text{low}} = \sum_{i=1}^{N-1} c_i(s)\Phi_i(s,z)$$

and the $\Phi_i(s,z)$ are uniformly bounded by Theorem 3.9 and Theorem 3.8. So it suffices to estimate $c_i(s)$. Recall that $c_i = 2\pi\dfrac{b_i}{\lambda_i}$ and that the small eigenvalues $\lambda_i(s)$ are well-studied by Dai-Yoshikawa in Theorem 2.1. It remains to determine the asymptotic properties of

$$b_i(s) = \int_{X_s} \Phi_i(s,z)\alpha$$

around 0 for all $1 \leq i \leq N$.

Our tool is the decomposition of Φ_i in Theorem 2.10. We recall the results here: For each $1 \leq i \leq N-1$, we have

$$\Phi_i(s,z) = \sum_{l=1}^{N} C_{li}(s)\Upsilon^{(l)}(s,z) + R_i(s,z).$$

And

- There is a uniform constant C such that $|C_{li}(s)| < C$ for all $1 \leq l \leq N$ and all s.
- The model functions $\Upsilon^{(l)}(s,z)$ are discussed in Proposition 2.8.
- The remainder terms R_i are discussed in Theorem 2.10.

Then each b_i is decomposed into

$$b_i(s) = \sum_{l=1}^{N} C_{li}(s)\int_{X_s}\Upsilon_s^{(l)}\alpha + \int_{X_s} R_i(s,z)\alpha. \tag{3.5}$$

We have the following lemma

Lemma 3.11. *Under the preceding notation. For any smooth* $(1,1)$*-form* α *on* X*,*

$$\int_{X_s}\Upsilon_s^{(l)}\alpha = \frac{1}{\sqrt{\text{Area}(C_l)}}\Big(\int_{C_l}\alpha + \|\alpha\|_\infty O(|s|^{\frac{1}{4\nu}})\Big).$$

Proof. (of Lemma 3.11) From the construction of $\Upsilon_s^{(l)}$ in Proposition 2.8, it follows that

$$\int_{X_s}\Upsilon_s^{(l)}\alpha = \int_{X_0}\frac{\chi_{\epsilon(s)}^{(l)}F_s^*\alpha}{\sqrt{\text{Area}(C_l)}},$$

where $\epsilon(s) = 2\,|s|^{\frac{1}{8\nu}}$.

Since α is a smooth form on X, by Theorem 2.4(4) and Remark 2.5, we have

$$\|F_s^*\alpha - \alpha_0\|_{L^\infty(X_0\setminus\cup_{p\in\text{Sing}\,X_0}B(p,\epsilon(s)))} \leq K_2\,|s|^{\frac{1}{2}}$$

for a uniform constant $K_2 > 0$. So

$$\left|\int_{X_s}\Upsilon_s^{(l)}\alpha - \int_{X_0}\frac{\chi_{\epsilon(s)}^{(l)}\alpha}{\sqrt{\text{Area}(C_l)}}\right| \leq K_2\sqrt{\text{Area}(C_l)}\,|s|^{\frac{1}{2}}. \tag{3.6}$$

Since $\chi_{\epsilon(s)}^{(l)}$ is compactly supported on C_l, $\left|\chi_{\epsilon(s)}^{(l)}-1\right|$ is bounded by 2 and it vanishes on C_l except on a set of measure $O(|s|^{\frac{1}{4\nu}})$ (by Lemma 2.7) , we have

$$\left|\int_{X_0}\frac{\chi_{\epsilon(s)}^{(l)}\alpha}{\sqrt{\operatorname{Area}(C_l)}}-\int_{C_l}\frac{\alpha}{\sqrt{\operatorname{Area}(C_l)}}\right|=\|\alpha\|_\infty O(|s|^{\frac{1}{4\nu}}). \tag{3.7}$$

Combining Equation (3.6) and Equation (3.7), we obtain

$$\left|\int_{X_s}\Upsilon_s^{(l)}\alpha-\frac{1}{\sqrt{\operatorname{Area}(C_l)}}\int_{C_l}\alpha\right|=\|\alpha\|_\infty O(|s|^{\frac{1}{4\nu}}).$$

□

On the other hand, from Theorem 2.10(2), we have

$$\left|\int_{X_s}R_i(s,z)\alpha\right|=\|\mathrm{tr}_\omega\alpha\|_{L^2(X_s)}O(\log^{-\frac{1}{2}}(|s|^{-1})).$$

Putting all these asymptotic results into Equation (3.5), we have

$$\begin{aligned}b_i(s)&=\sum_{l=1}^N\frac{C_{li}(s)}{\sqrt{\operatorname{Area}(C_l)}}\int_{C_l}\alpha+\sum_{l=1}^N\frac{C_{li}(s)}{\sqrt{\operatorname{Area}(C_l)}}O(|s|^{\frac{1}{4\nu}})\\&\quad+\|\mathrm{tr}_\omega\alpha\|_{L^2(X_s)}O(\log^{-\frac{1}{2}}(|s|^{-1}))\\&=\sum_{l=1}^N\frac{C_{li}(s)}{\sqrt{\operatorname{Area}(C_l)}}\int_{C_l}\alpha+O(\log^{-\frac{1}{2}}(|s|^{-1}))\\&=O(1)\|\mathrm{tr}_{\omega_0}\alpha\|_{L^1(X_0)}+O(\log^{-\frac{1}{2}}(|s|^{-1}))\|\mathrm{tr}_\omega\alpha\|_{L^2(X_s)}.\end{aligned} \tag{3.8}$$

We then prove the second part of Theorem 3.4.

Proof. (of Theorem 3.4.(2)) For α whose integral is zero on general fibers, following Equation (3.8), we have $b_i(s)=O(\log^{-\frac{1}{2}}(|s|^{-1}))\|\mathrm{tr}_\omega\alpha\|_{L^2(X_s)}$.

Since $\varphi_{\mathrm{low}}=\sum_{i=1}^{N-1}c_i(s)\Phi_i(s,z)$ and $c_i=2\pi b_i/\lambda_i$, we have

$$c_i(s)=O(\log(|s|^{-1}))\|\mathrm{tr}_{\omega_0}\alpha\|_{L^1(X_0)}+O(\log^{\frac{1}{2}}(|s|^{-1}))\|\mathrm{tr}_\omega\alpha\|_{L^2(X_s)}.$$

Since $\Phi_i(s,z)$ is uniformly bounded for $1\le i\le N-1$, we have

$$\|\varphi_{\mathrm{low}}\|_{L^\infty(X_s)}\le C\|\mathrm{tr}_{\omega_0}\alpha\|_{L^1(X_0)}\log(|s|^{-1}).$$

□

3.C. **Estimate of the low frequency part (II)**

If α additionally satisfies Condition 3.2 on the singular fiber (i.e., α has zero integral on each irreducible component of the singular fiber), we obtain a better estimate on its preferred potential (Theorem 3.5). We prove the theorem here.

Following the same strategy as in Section 3.B, it suffices to estimate each term in the expansion Equation (3.5) of $b_i(s)$:

$$b_i(s) = \sum_{l=1}^{N} C_{li}(s) \int_{X_s} \Upsilon_s^{(l)}\alpha + \int_{X_s} R_i(s,z)\alpha.$$

The first part is good enough when α has zero integral on each irreducible component of the singular fiber as proved in Lemma 3.11, and we have

$$\sum_{l=1}^{N} C_{li}(s) \int_{X_s} \Upsilon_s^{(l)}\alpha = O(|s|^{\frac{1}{4\nu}}).$$

It remains to estimate $\int_{X_s} R_i(s,z)\alpha$ under conditions Condition 3.2, which is the following proposition.

Proposition 3.12. *For $s \neq 0$, we define $\widetilde{R}_i \coloneqq \log^{\frac{1}{2}}(|s|^{-1})R_i$. If α is a smooth $(1,1)$-form on X with Condition 3.2, we then have*

$$\log^{\frac{1}{2}}(|s|^{-1}) \left| \int_{X_s} R_i(s,z)\alpha \right| = \left| \int_{X_s} \widetilde{R}_i(s,z)\alpha \right| \to 0 \quad \textit{as } s \to 0.$$

Proof. We prove it by contradiction. Suppose we have a sequence $s_n \to 0$ such that $\left|\int_{X_{s_n}} \widetilde{R}_i(s_n,z)\alpha\right| \geq \epsilon_0$ for some $\epsilon_0 > 0$.

First, we check that the family of functions $\widetilde{R}_{i,s}(z) = \widetilde{R}_i(s,z) \in C^\infty(X_s)$ satisfies the conditions in Proposition 1.13. We denote $\mathrm{dd}^c_z \widetilde{R}_{i,s}(z)$ by β_s.

- For the first condition in Proposition 1.13, we use the decomposition formula for eigenfunctions Φ_i in Equation (2.2):

$$\Phi_i(s,z) = \sum_{l=1}^{N} C_{l(i+1)}(s)\Upsilon^{(l)}(s,z) + R_i(s,z).$$

  Around any $x \in X_{0,\mathrm{reg}}$, we can take an open neighborhood U of x such that $\Upsilon^{(l)}(s,z)$ is constant on U by Corollary 2.9. Then on $U \cap X_s$, we have

$$\mathrm{dd}^c_z \widetilde{R}_{i,s}(z) = \beta_s = \mathrm{dd}^c_z \Phi_i(s,z) = -\lambda_i(s)\Phi_i(s,z)\omega_s.$$

  So $\|\mathrm{tr}_{\omega_s}\beta_s\|_{L^\infty(U\cap X_s)} = \|\lambda_i(s)\Phi_i(s,z)\|_{L^\infty(U\cap X_s)}$ is uniformly bounded in s.
- For the second condition in Proposition 1.13, we note that

$$\|\mathrm{tr}_{\omega_s}\beta_s\|_{C^0(U\cap X_s)} = \lambda_i(s)\,\|\Phi_i\|_{C^0(U\cap X_s)} \to 0.$$

  So $\mathrm{tr}_{\omega_s}\beta_s \to 0 = f_0$ in C^0_{loc} on $X_{0,\mathrm{reg}}$.
- For the third condition in Proposition 1.13, it is proved in Theorem 2.10(3) that there is a uniform constant $C > 0$ such that

$$\|\mathrm{d}R_j\|^2_{L^2(X_s)} \leq \frac{C^2}{\log(|s|^{-1})}, \quad \|R_j\|^2_{L^2(X_s)} \leq C^2 \frac{1}{\log(|s|^{-1})}.$$

  for all $1 \leq j \leq N$ and all s. So

$$\left\|\widetilde{R}_{i,s}(z)\right\|_{L^2(X_s,\omega_s)}, \left\|\mathrm{d}\widetilde{R}_{i,s}(z)\right\|_{L^2(X_s,\omega_s)} \leq C^2.$$

We then apply Proposition 1.13 to the family of functions $\widetilde{R}_{i,s}(z)$ and sequence $s_n \to 0$, we then have a subsequence s_{k_n} such that

- $\widetilde{R}_{i,s_{k_n}}(z)$ converges to $\widetilde{R}_0(z)$ in $C^{1,a}_{\mathrm{loc}}$ on $X_{0,\mathrm{reg}}$ for any $0 < a < 1$,
- $\widetilde{R}_0(z) \in W^{2,p}_{\mathrm{loc}}$ for all $1 < p < \infty$ and it solves $-\Delta_{\omega_0}\widetilde{R}_0(z) = f_0 = 0$ a.e. on $X_{0,\mathrm{reg}}$,
- $\widetilde{R}_0(z) \in W^{1,2}(X_{0,\mathrm{reg}})$,
- $\int_{X_{s_{k_n}}} \widetilde{R}_{i,s_{k_n}}(z)\alpha \to \int_{X_0} \widetilde{R}_0(z)\alpha$.

Since $\widetilde{R}_0(z) \in W^{1,2}(X_0)$ and it is harmonic on $X_{0,\mathrm{reg}}$, it is locally constant and is constant on each irreducible component of X_0. So we have $\int_{X_0} \widetilde{R}_0(z)\alpha = 0$. Thus the initial assumption $\left|\int_{X_{s_n}} \widetilde{R}_i(s_n, z)\alpha\right| \geq \epsilon_0$ for all n and an $\epsilon_0 > 0$ contradicts that

$$\int_{X_{s_{k_n}}} \widetilde{R}_{i,s_{k_n}}(z)\alpha \to \int_{X_0} \widetilde{R}_0(z)\alpha = 0.$$

This completes the proof. □

As a direct consequence, we can prove Theorem 3.5

Proof. (of Theorem 3.5) Following previous discussion, we have

$$b_i(s) = O(|s|^{\frac{1}{4\nu}}) + o(\log^{-\frac{1}{2}}(|s|^{-1})).$$

So $c_i(s) = 2\pi\frac{b_i(s)}{\lambda_i(s)} = o(\log^{\frac{1}{2}}(|s|^{-1}))$ for $1 \leq i \leq N-1$. It implies that

$$\frac{\|\varphi_{\mathrm{low}}\|_{L^\infty(X_s)}}{\log^{1/2}(|s|^{-1})} \to 0, \quad \text{as } s \to 0.$$

□

4. Asymptotics of Archimedean height pairing

As an application of previous estimates, we introduce *the Archimedean height pairing for smooth real* $(1,1)$*-forms on* X. We continue to use the setup from Setup 1, except in Corollary 4.6, where we allow for general singular fibers of the degeneration.

Definition 4.1 (Archimedean height pairing)**.** *Let* α, β *be two smooth real* $(1,1)$*-forms on* X *with* $\int_{X_s}\alpha = \int_{X_s}\beta = 0$ *for* $s \neq 0$ *(Condition 3.1). Let* φ *be the preferred potential of* α*. We define* the Archimedean height pairing $\langle\alpha,\beta\rangle$ *to be a function on* $\mathbb{D}^\circ$

$$\langle\alpha,\beta\rangle(s) := \int_{X_s}\varphi\beta, \quad s \in \mathbb{D}^\circ.$$

It is smooth on $\mathbb{D}^\circ$ *since* $\pi\colon X^\circ(:= X \setminus X_0) \to \mathbb{D}^\circ$ *is a submersion and* φ *is smooth.*

Remark 4.2. We can use any potential for α on X_s to define the Archimedean height pairing. Actually, for any potential $-\varphi'$ of α on X_s(i.e., $\mathrm{dd}^c\varphi' + \alpha = 0$ on X_s), the difference $\varphi' - \varphi$ is constant on X_s, where φ is the preferred potential of α. So $\int_{X_s}(\varphi'-\varphi)\beta = \text{constant}\cdot\int_{X_s}\beta = 0$ and $\int_{X_s}\varphi'\beta = \int_{X_s}\varphi\beta = \langle\alpha,\beta\rangle(s)$.

We now state some basic properties of the Archimedean height pairing defined above.

Proposition 4.3. *Let* α, β *be two forms as in Definition 4.1. We have*

(1) The Archimedean height pairing is symmetric and bilinear.
(2) The Archimedean height pairing is positive definite, i.e., $\langle \alpha, \alpha \rangle$ is a non-negative function on $\mathbb{D}^\circ$, and $\langle \alpha, \alpha \rangle = 0$ if and only if $\alpha\mid_{X_s} = 0$ for $s \neq 0$.

Proof.
(1) Since the preferred potentials depend linearly on the differential forms, we have that the Archimedean height pairing is bilinear. It is symmetric because $\langle \alpha, \beta \rangle(s) = \int_{X_s} \varphi(-\mathrm{dd}^c \varphi') = \int_{X_s} \varphi'(-\mathrm{dd}^c \varphi) = \langle \beta, \alpha \rangle(s)$, where φ, φ' are preferred potentials for α, β respectively.
(2) We have $\langle \alpha, \alpha \rangle(s) = \int_{X_s} \varphi(-\mathrm{dd}^c \varphi) = \frac{1}{4\pi} \|\mathrm{d}\varphi\|^2_{L^2(X_s)} \geq 0$. If $\langle \alpha, \alpha \rangle = 0$, then the preferred potential φ of α is fiberwise constant, so $\alpha|_{X_s} = 0$ for $s \neq 0$.

□

Theorem 4.4 (Continuity of pairing). *Let α, β be two smooth real $(1,1)$-forms on X with Condition 3.1 satisfying*

$$\int_{C_i} \alpha = \int_{C_i} \beta = 0,$$

where the C_i are the irreducible components of the reduced singular fiber X_0 (Condition 3.2).

Then the pairing function $\langle \alpha, \beta \rangle \in C^0(\mathbb{D}^\circ)$ (in Definition 4.1) can be continuously extended to $\mathbb{D}$.

Proof. (of Theorem 4.4) Let $\varphi, \rho \in C^\infty(X \setminus X_0)$ be preferred potentials of α and β respectively.

By Lemma 2.3 and shrinking the base, we decompose φ into its low frequency part $\varphi_{\text{low}} \in C^\infty(X \setminus X_0)$ and high frequency part $\varphi_{\text{high}} \in C^\infty(X \setminus X_0)$. Then

$$\begin{aligned} \langle \alpha, \beta \rangle(s) &= -\int_{X_s} (\varphi_{\text{low}} + \varphi_{\text{high}}) \mathrm{dd}^c \rho \\ &= -\int_{X_s} \rho \mathrm{dd}^c \varphi_{\text{low}} - \int_{X_s} \varphi_{\text{high}} \beta \end{aligned}$$

Using the asymptotic results from Theorem 3.5, we have $\rho = o(\log^{\frac{1}{2}}(|s|^{-1}))$ and $\varphi_{\text{low}} = o(\log^{\frac{1}{2}}(|s|^{-1}))$. Since φ_{low} is in the space spanned by eigenfunctions with low frequency, we have

$$2\pi \|\mathrm{tr}_{\omega_s} \mathrm{dd}^c \varphi_{\text{low}}\|_{L^\infty(X_s)} = \|\Delta_{\omega_s} \varphi_{\text{low}}\|_{L^\infty(X_s)} = o(\log^{-\frac{1}{2}}(|s|^{-1})).$$

So $\int_{X_s} \rho \mathrm{dd}^c \varphi_{\text{low}} = o(1)$ and it is continuous at 0.

To show that $\int_{X_s} \varphi_{\text{high}} \beta$ can be continuously extended to $\mathbb{D}$, we use Corollary 1.14. It suffices to check that $\varphi_{\text{high,s}} \in C^\infty(X_s)$ satisfies the conditions in Proposition 1.13. Note that

$$\mathrm{dd}^c_z \varphi_{\text{high,s}}(z) = -\alpha - \mathrm{dd}^c_z \varphi_{\text{low,s}}(z)$$

- For the first condition, we have

$$\begin{aligned} \|\mathrm{tr}_{\omega_s} \mathrm{dd}^c_z \varphi_{\text{high,s}}(z)\|_{L^\infty(X_s)} &\leq \|\mathrm{tr}_{\omega_s} \alpha\|_{L^\infty(X_s)} + \|\mathrm{tr}_{\omega_s} \mathrm{dd}^c_z \varphi_{\text{low,s}}\|_{L^\infty(X_s)} \\ &= \|\mathrm{tr}_{\omega_s} \alpha\|_{L^\infty(X_s)} + \frac{1}{2\pi} \|\Delta_{\omega_s} \varphi_{\text{low}}\|_{L^\infty(X_s)} \\ &= O(1) + o(\log^{-\frac{1}{2}}(|s|^{-1})) \end{aligned}$$

So $\|\mathrm{tr}_{\omega_s}\mathrm{dd}^c_z\varphi_{\mathrm{high,s}}(z)\|_{L^\infty(X_s)}$ is uniformly bounded in s.

- Since $\mathrm{tr}_{\omega_s}\alpha$ is continuous on $X\setminus\mathrm{Sing}\,X_0$ and
$$\|\mathrm{tr}_{\omega_s}\mathrm{dd}^c_z\varphi_{\mathrm{high,s}}(z)-\mathrm{tr}_{\omega_s}\alpha\|_{L^\infty(X_s)}=\frac{1}{2\pi}\|\Delta_{\omega_s}\varphi_{\mathrm{low,s}}\|_{L^\infty(X_s)}=o(\log^{-\frac{1}{2}}(|s|^{-1})).$$
We have $\mathrm{tr}_{\omega_s}\mathrm{dd}^c_z\varphi_{\mathrm{high,s}}\to\mathrm{tr}_{\omega_0}\alpha$ in C^0_{loc} on $X_{0,\mathrm{reg}}$, which is the second condition.
- Since $\|\varphi_{\mathrm{high}}\|_{L^\infty(X_s)}$ (by Theorem 3.4) and $\|\mathrm{tr}_{\omega_s}\mathrm{dd}^c_z\varphi_{\mathrm{high,s}}(z)\|_{L^\infty(X_s)}$ are uniformly bounded, $\|\varphi_{\mathrm{high}}\|_{L^2(X_s)}$ and $\|\mathrm{d}\varphi_{\mathrm{high}}\|_{L^2(X_s)}$ are also uniformly bounded. So the third condition is verified.

So we can apply Corollary 1.14 to $\varphi_{\mathrm{high,s}}$ and $\int_{X_s}\varphi_{\mathrm{high}}\beta$ is then continuous at 0. In summary, $\langle\alpha,\beta\rangle(s)$ can be continuously extended to $\mathbb{D}$. ☐

For general smooth classes α and β satisfying Condition 3.1, we can extend Theorem 4.4.

Theorem 4.5 (Asymptotics of pairing)**.** *Let α and β be two smooth real $(1,1)$-forms whose integrals on general fibers are* 0*. Then there is a constant $c_{\alpha,\beta}$ such that*
$$\langle\alpha,\beta\rangle-c_{\alpha,\beta}\log|s|^2$$
extends continuously to $\mathbb{D}$.

The constant $c_{\alpha,\beta}$ is determined in the following way: Let M be the intersection matrix of irreducible components of the singular fiber X_0, given by $M_{ij}=C_i\cdot C_j$. Let M^+ be the Moore-Penrose pseudoinverse of M. Then $c_{\alpha,\beta}=\mathbf{v}_\alpha^T(M^+)^T\mathbf{v}_\beta$, where
$$\mathbf{v}_\alpha=\left(\int_{C_1}\alpha,\dots,\int_{C_N}\alpha\right)^T,\quad\mathbf{v}_\beta=\left(\int_{C_1}\beta,\dots,\int_{C_N}\beta\right)^T.$$

In particular, if either α or β satisfies the integrability condition on the singular fiber *(Condition 3.2), then $c_{\alpha,\beta}=0$ because either $\mathbf{v}_\alpha$ or $\mathbf{v}_\beta$ is zero.*

Proof. Since the singular fiber is reduced, we have $X_0=\sum_{i=1}^N C_i$, where C_i are the irreducible components.

We consider the $N\times N$ intersection matrix M with entries $M_{ij}=C_i\cdot C_j$. By Zariski's Lemma (see [BHPVdV04, III.8.2, p.111]), we have

- The matrix M is negative semi-definite.
- The kernel of M is exactly one-dimensional and is spanned by $\mathbf{1}=(1,1,\cdots,1)$.

Let $V_0=\left\{\mathbf{x}\in\mathbb{R}^N:\ \sum_{i=1}^N x_i=0\right\}$ be the degree-zero subspace. Then $M|_{V_0}$ is strictly negative definite and is invertible. To compute the inverse of $M|_{V_0}$, we introduce M^+ the Moore-Penrose pseudoinverse of M. Then M^+ acts as the inverse of M on V_0 and annihilates $\mathrm{span}(\mathbf{1})$.

For each C_i, the irreducible component of X_0, we consider the line bundle $L_i=\mathcal{O}(C_i)$ equipped with a Hermitian metric h_i. Then we have a closed smooth $(1,1)$-form $\gamma_i=-\mathrm{dd}^c\log\|e\|^2_{h_i}$ representing the class $[C_i]$, where e is a local frame of the holomorphic line bundle L_i. Let σ_i be the canonical section of L_i which vanishes exactly on C_i. Then the preferred potential for γ_i on each smooth fiber X_s is $\log\|\sigma_i\|^2_{h_i}$ up to a constant.

We define the following map, which maps onto differential forms representing cohomology classes spanned by $\{[C_i]\}$:

$$\text{Form}\colon \mathbb{R}^N \to \{\text{smooth closed } (1,1)\text{-forms with zero integral on the singular fiber}\}$$

$$\mathbf{x} = (x_1, \cdots, x_N) \mapsto \sum_{i=1}^N x_i \gamma_i =: \text{Form}(\mathbf{x})$$

Furthermore, the preferred potential of $\text{Form}(\mathbf{x})$ on X_s will be $\sum_{i=1}^N x_i \log \|\sigma_i\|_{h_i}^2$ up to a constant depending on $s \neq 0$.

For smooth $(1,1)$-forms α, β on X such that $\int_{X_s} \alpha = \int_{X_s} \beta = 0$ for $s \neq 0$, the vectors of component integrals

$$\mathbf{v}_\alpha = \left(\int_{C_1} \alpha, \ldots, \int_{C_N} \alpha \right)^T, \quad \mathbf{v}_\beta = \left(\int_{C_1} \beta, \ldots, \int_{C_N} \beta \right)^T.$$

are in V_0. We can then decompose α, β into

$$\alpha = \text{Form}(M^+\mathbf{v}_\alpha) + \underbrace{(\alpha - \text{Form}(M^+\mathbf{v}_\alpha))}_{=:\alpha_0}$$

$$\beta = \text{Form}(M^+\mathbf{v}_\beta) + \underbrace{(\beta - \text{Form}(M^+\mathbf{v}_\beta))}_{=:\beta_0}$$

Note that for each C_i,

$$\begin{aligned} \int_{C_i} \alpha_0 &= \text{i-th coordinate of } (\mathbf{v}_\alpha - MM^+\mathbf{v}_\alpha) \\ &= 0 \quad (\mathbf{v}_\alpha \in V_0, \text{ so } MM^+\mathbf{v}_\alpha = \mathbf{v}_\alpha). \end{aligned}$$

Similarly, we have $\int_{C_i} \beta_0 = 0$ for each C_i.

Since the pairing $\langle \alpha, \beta \rangle$ is bilinear and symmetric, we have

$$\begin{aligned} \langle \alpha, \beta \rangle = &\langle \alpha_0, \beta_0 \rangle + \langle \alpha_0, \text{Form}(M^+\mathbf{v}_\beta) \rangle + \langle \text{Form}(M^+\mathbf{v}_\alpha), \beta_0 \rangle \\ &+ \langle \text{Form}(M^+\mathbf{v}_\alpha), \text{Form}(M^+\mathbf{v}_\beta) \rangle \end{aligned}$$

By the previous theorem (Theorem 4.4), $\langle \alpha_0, \beta_0 \rangle$ can be extended continuously to S.

Since the preferred potential for $\text{Form}(M^+\mathbf{v}_\alpha)$ is $\sum_{i=1}^N (M^+\mathbf{v}_\alpha)_i \log \|\sigma_i\|_{h_i}^2$ up to a constant on X_s, we have

$$\langle \text{Form}(M^+\mathbf{v}_\alpha), \beta_0 \rangle = \sum_{i=1}^N \int_{X_s} (M^+\mathbf{v}_\alpha)_i \log \|\sigma_i\|_{h_i}^2 \beta_0$$

Using the asymptotics of log-norm integral (Lemma A.1), each $\int_{X_s} \log \|\sigma_i\|_{h_i}^2 \beta_0$ extends continuously on S. Thus, $\langle \text{Form}(M^+\mathbf{v}_\alpha), \beta_0 \rangle$ extends Hölder continuously to S.

Similarly, $\langle \alpha_0, \text{Form}(M^+\mathbf{v}_\beta) \rangle$ extends Hölder continuously to S.

For the last term, we have

$$\langle \text{Form}(M^+\mathbf{v}_\alpha), \text{Form}(M^+\mathbf{v}_\beta) \rangle = \sum_{i=1}^N \sum_{j=1}^N (M^+\mathbf{v}_\alpha)_i (M^+\mathbf{v}_\beta)_j \int_{X_s} \log \|\sigma_i\|_{h_i}^2 \gamma_j.$$

By Lemma A.1, $\int_{X_s} \log\|\sigma_i\|^2_{h_i}\gamma_j - \int_{C_i}\gamma_j \log|s|^2$ can be extended Hölder continuously through the origin, so

$$\langle \mathrm{Form}(M^+\mathbf{v}_\alpha), \mathrm{Form}(M^+\mathbf{v}_\beta)\rangle - \underbrace{\left(\sum_{1\le i,j\le N}(M^+\mathbf{v}_\alpha)_i(M^+\mathbf{v}_\beta)_j\int_{C_i}\gamma_j\right)}_{=:c_{\alpha,\beta}}\log|s|^2$$

extends Hölder continuously on S. Here $c_{\alpha,\beta}$ can be simplified as

$$\begin{aligned} c_{\alpha,\beta} = \sum_{1\le i,j\le N}(M^+\mathbf{v}_\alpha)_i(M^+\mathbf{v}_\beta)_j M_{ij} &= (M^+\mathbf{v}_\alpha)^T M M^+\mathbf{v}_\beta \\ &= (M^+\mathbf{v}_\alpha)^T v_\beta \end{aligned}$$

In summary, $\langle\alpha,\beta\rangle - c_{\alpha,\beta}\log|s|^2$ extends continuously to $\mathbb{D}$. □

We note that the height pairing $\langle\alpha,\beta\rangle$ is well-behaved under semi-stable reduction. So we can extend the asymptotic result in Theorem 4.5 to the degeneration of algebraic curves with non-reduced singular fibers.

Corollary 4.6. *Let $\pi\colon X\to\mathbb{D}$ be a degeneration of algebraic curves. Assume X is smooth, and allow the singular fiber X_0 to be non-reduced and non-irreducible. Let $p\colon Y\to\mathbb{D}$ be a semi-stable reduction of (X, X_0) with the following commutative diagram*

$$\begin{array}{ccc} Y & \xrightarrow{\mu} & X \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \pi} \\ \mathbb{D}_t & \xrightarrow{f_d} & \mathbb{D}_s \end{array}$$

Then $f_d(t)=t^d$ and Y_0 is a reduced divisor with simple normal crossings. Furthermore, Y is a Kähler manifold.

Let α,β be two smooth real $(1,1)$-forms on X with Condition 3.1. Then $\mu^*\alpha$ and $\mu^*\beta$ are two smooth $(1,1)$-forms on Y with Condition 3.1. So we can define $\langle\alpha,\beta\rangle\in C^\infty(\mathbb{D}_s^\circ)$ and $\langle\mu^*\alpha,\mu^*\beta\rangle\in C^\infty(\mathbb{D}_t^\circ)$. These two pairings are related by

$$\langle\alpha,\beta\rangle\circ f_d = \langle\mu^*\alpha,\mu^*\beta\rangle. \tag{4.1}$$

In particular, there is a constant $c_{\alpha,\beta}\in\mathbb{R}$ such that $\langle\alpha,\beta\rangle - c_{\alpha,\beta}\log|s|^2$ extends continuously to $\mathbb{D}$. So $\langle\alpha,\beta\rangle\in L^\infty(\mathbb{D}^\circ)$ if and only if $\langle\alpha,\beta\rangle\in C^0(\mathbb{D})$.

Proof. It suffices to prove Equation (4.1).

For $0\ne t_0\in\mathbb{D}_t$, we have an isomorphism $\mu_{t_0}\colon Y_{t_0}\to X_{f_d(t_0)}$. Let $-\varphi$ be a potential for α on $X_{f_d(t_0)}$, i.e., $\mathrm{dd}^c\varphi+\alpha=0$ on $X_{f_d(t_0)}$. Then $-\mu^*\varphi$ is a potential for $\mu^*\alpha$ on Y_{t_0}. So

$$\langle\mu^*\alpha,\mu^*\beta\rangle(t_0) = \int_{Y_{t_0}}(\mu^*\varphi)\mu^*\beta = \int_{X_{f_d(t_0)}}\varphi\beta = \langle\alpha,\beta\rangle(f_d(t_0)).$$

Thus $\langle\alpha,\beta\rangle\circ f_d = \langle\mu^*\alpha,\mu^*\beta\rangle$ on $\mathbb{D}_t^\circ$.

By Theorem 4.5, $\langle\mu^*\alpha,\mu^*\beta\rangle - c_{\mu^*\alpha,\mu^*\beta}\log|t|^2$ is continuous on $\mathbb{D}_t$. So we can find $c_{\alpha,\beta}\in\mathbb{R}$ depending on $c_{\mu^*\alpha,\mu^*\beta}$ and d such that $\langle\alpha,\beta\rangle - c_{\alpha,\beta}\log|s|^2$ is continuous on $\mathbb{D}_s$. □

At the end of this section, we propose a question on the Hölder continuity of the pairing.

Question 4.7. *We consider a degeneration of algebraic curves with reduced singular fibers as in Setup 1. Let α, β be two smooth real $(1,1)$-forms on X satisfying $\int_{C_i} \alpha = \int_{C_i} \beta = 0$, where C_i are irreducible components of the reduced singular fiber X_0. Is the pairing $\langle \alpha, \beta \rangle$ a Hölder continuous function?*

If this question is answered positively, we can strengthen the results in Theorem 4.5, Corollary 4.6, Proposition 5.4 and Corollary 5.5 by replacing the word "continuously" *with* "Hölder continuously"*.*

5. Application to K3 surfaces with parabolic automorphisms

As an application of our previous results, we study the parabolic dynamics of elliptic K3 surfaces, following [FT23, Section 3].

Let $\pi\colon X \to \mathbb{P}^1 =: B$ be a projective elliptic K3 surface. A *parabolic automorphism* T of X is an automorphism of X which preserves the fibration π and is of infinite order. This yields the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "\pi"'] & X \arrow[dl, "\pi"] \\ B := \mathbb{P}^1 & \end{tikzcd}$$

In some literature (e.g. [Fil22, 6.2.2]), *a parabolic automorphism* is also called a *twist automorphism*.

Compared to [FT23], we allow possibly non-irreducible and non-reduced singular fibers of $\pi\colon X \to B$ in the setup of this section.

We use B° to denote the smooth locus of the fibration π, and we use s to denote the local coordinate on B. We use $X_s := \pi^{-1}(s)$.

5.A. **Preliminaries**

Before delving deeper, we introduce basic notions and lemmas in this part.

Let $[E] \in \mathrm{NS}(X)$ be the class of a general fiber of π in the Néron–Severi group of X. We consider the group $\mathrm{Aut}_\pi(X)$ consisting of automorphisms of X preserving the elliptic fibration. The following filtration of $\mathrm{NS}(X)$ is preserved by $\mathrm{Aut}_\pi(X)$:

$$0 \subsetneq [E] \subsetneq [E]^\perp \subsetneq \mathrm{NS}(X),$$

where $[E]^\perp$ is the orthogonal complement of $[E]$ with respect to the intersection form in $\mathrm{NS}(X)$. By the Hodge index theorem, the intersection form on the quotient lattice $L := [E]^\perp/[E]$ is strictly negative definite.

Let $\mathrm{Isom}(L)$ be the group of automorphisms of the lattice L that preserve the induced intersection form. Since the induced intersection form is strictly negative-definite, $\mathrm{Isom}(L)$ is a discrete subgroup of the compact orthogonal group $O(\text{rank }(L))$. Thus, $\mathrm{Isom}(L)$ is finite.

Since $\mathrm{Aut}_\pi(X)$ preserves the lattice L equipped with the intersection form, we have the induced group homomorphism:

$$\rho\colon \mathrm{Aut}_\pi(X) \to \mathrm{Isom}(L).$$

We define $\ker(\rho) =: \mathbf{Aut}^\circ_{\boldsymbol{\pi}}\boldsymbol{(X)}$, which is the subgroup of $\mathrm{Aut}_\pi(X)$ whose induced action on $[E]^\perp/[E]$ is trivial. Because $\mathrm{Isom}(L)$ is finite, $\mathrm{Aut}^\circ_\pi(X)$ is a finite index subgroup of $\mathrm{Aut}_\pi(X)$.

If we work inside $\mathrm{H}^{1,1}(X,\mathbb{R})$, we have a similar construction of $\mathrm{Aut}^\circ_\pi(X)$.

Lemma 5.1. *Let* $[E]^\perp_{\mathrm{H}^{1,1}}$ *denote the orthogonal complement of* $[E]$ *in* $\mathrm{H}^{1,1}(X,\mathbb{R})$*. Then the following filtration is preserved by* $\mathrm{Aut}_\pi(X)$*:*

$$0 \subsetneq [E]_\mathbb{R} \subsetneq [E]^\perp_{\mathrm{H}^{1,1}} \subsetneq \mathrm{H}^{1,1}(X,\mathbb{R}).$$

We define $\mathbf{Aut}^{\circ,(1,1)}_{\boldsymbol{\pi}}\boldsymbol{(X)}$ *as the subgroup of* $\mathrm{Aut}_\pi(X)$ *that acts trivially on the quotient space* $[E]^\perp_{\mathrm{H}^{1,1}}/[E]_\mathbb{R}$*. Then* $\mathrm{Aut}^{\circ,(1,1)}_\pi(X)$ *is a finite index subgroup of* $\mathrm{Aut}_\pi(X)$*.*

Proof. Since the intersection form is non-degenerate, we have the following orthogonal decomposition

$$\mathrm{H}^{1,1}(X,\mathbb{R}) = \mathrm{NS}(X)_\mathbb{R} \oplus T^{1,1}(X)_\mathbb{R},$$

where $T^{1,1}(X)_\mathbb{R}$ is the degree $(1,1)$-part of the real transcendental classes $T(X)_\mathbb{R}$, and $T(X)$ is the transcendental lattice of X (see [Huy16, Chapter 3, Lemma 3.1]).

Note that $[E] \in \mathrm{NS}(X)$, so it is perpendicular to transcendental classes $T^{1,1}(X)_\mathbb{R}$. Thus, $[E]^\perp_{\mathrm{H}^{1,1}}$ has a similar decomposition

$$[E]^\perp_{\mathrm{H}^{1,1}} = [E]^\perp_\mathbb{R} \oplus T^{1,1}(X)_\mathbb{R},$$

which descends to the quotient space as:

$$[E]^\perp_{\mathrm{H}^{1,1}}/[E]_\mathbb{R} = ([E]^\perp/[E])_\mathbb{R} \oplus T^{1,1}(X)_\mathbb{R}.$$

Since $\mathrm{Aut}_\pi(X)$ preserves the transcendental lattice $T(X)$ with its Hodge structure, the image of the induced group homomorphism:

$$\rho_T\colon \mathrm{Aut}_\pi(X) \to \mathrm{Isom}(T(X)_\mathbb{R})$$

is a finite group by [Huy16, Chapter 3, Corollary 3.4]. Thus $\ker(\rho_T)$ is a finite index subgroup of $\mathrm{Aut}_\pi(X)$.

Since $T^{1,1}(X)_\mathbb{R} \subset T(X)_\mathbb{R}$, we have

$$\ker(\rho_T) \cap \mathrm{Aut}^\circ_\pi(X) \subset \mathrm{Aut}^{\circ,(1,1)}_\pi(X).$$

Since $\ker(\rho_T)$ and $\mathrm{Aut}^\circ_\pi(X)$ are finite index subgroups, $\mathrm{Aut}^{\circ,(1,1)}_\pi(X)$ is also a finite index subgroup of $\mathrm{Aut}_\pi(X)$. ☐

Remark 5.2. Lemma 5.1 also holds for general projective elliptic surfaces. To see this, it suffices to show that the image of the induced homomorphism $\rho_T\colon \mathrm{Aut}_\pi(X) \to \mathrm{Isom}(T(X)_\mathbb{R})$ is a finite group. This follows from the fact that $\mathrm{Aut}_\pi(X)$ preserves the integral lattice $T(X)$ along with its Hodge structure, and the intersection form on the $(1,1)$-part $T^{1,1}(X)_\mathbb{R}$ is strictly negative definite. (See also the proof of [Huy16, Chapter 3, Corollary 3.4].)

Let r be the index of $\mathrm{Aut}_\pi^{\circ,(1,1)}(X)$ inside $\mathrm{Aut}_\pi(X)$. By elementary group theory, we have $T^{r!} \in \mathrm{Aut}_\pi^{\circ,(1,1)}(X)$ for any $T \in \mathrm{Aut}_\pi(X)$.

Lastly, we introduce a useful formula on cohomology classes, which follows directly from the definition of $\mathrm{Aut}_\pi^{\circ,(1,1)}(X)$.

Lemma 5.3. *Let v be a real $(1,1)$-class in the cohomology group $\mathrm{H}^{1,1}(X)$. Suppose $v \cdot [E] = 0$. Then for any $T \in \mathrm{Aut}_\pi^{\circ,(1,1)}(X)$, we have*

$$T_* v - v = c(v,T)[E],$$

where $c(v,T) \in \mathbb{R}$ depends linearly on v.

5.B. Current-valued pairing

We recall the construction of *current-valued pairing* from [FT23, Theorem 3.2.14]. We relate it to the *Archimedean height pairing of smooth forms* in Definition 4.1.

Let α be a closed smooth real $(1,1)$-form on X such that $\int_{X_s} \alpha = 0$ for $s \neq 0$. Let $\varphi \in C^\infty(X^\circ)$ be the *preferred potential* of α.

Cohomologically, we have $[\alpha] \cdot [E] = 0$. Using Lemma 5.3, we obtain

$$T_*[\alpha] - [\alpha] = c[E],$$

for $T \in \mathrm{Aut}_\pi^{\circ,(1,1)}(X)$.

Note that $[E]$ can be represented by the smooth form $\pi^*\beta$, where β is a suitable smooth form on the base $B = \mathbb{P}^1$. We can take an $f \in C^\infty(X)$ such that

$$T_*\alpha - \alpha = \pi^*\beta + \mathrm{dd}^c f. \tag{5.1}$$

Then $f + T_*\varphi - \varphi$ is constant on each X_s for $s \in B^\circ$. Therefore, there is a function u on B° such that

$$\pi^* u = f + T_*\varphi - \varphi. \tag{5.2}$$

Let ω be a Kähler metric on X normalized such that $[\omega] \cdot [E] = 1$. Then $u(s)$ can be determined by

$$\int_{X_s} (\pi^* u)\omega_s = u(s) = \int_{X_s} (f + T_*\varphi - \varphi)\omega_s \tag{5.3}$$

We call u the *limit potential.* We rewrite Equation (5.3) using the *Archimedean height pairing* in Definition 4.1:

$$u(s) = \int_{X_s} (f + T_*\varphi - \varphi)\omega_s = \int_{X_s} f\omega_s + \langle \alpha, (T^* - I)\omega \rangle \tag{5.4}$$

The *current-valued pairing* is defined as $\eta_B(T, [\alpha]) := \beta + \mathrm{dd}^c u$, which is a current on B. It is smooth on B° since $u \in C^\infty(B^\circ)$.

We now generalize Filip–Tosatti's current-valued pairing by allowing arbitrary singular fibers of the elliptic fibration $\pi\colon X \to B$.

Proposition 5.4. *Let $T \in \mathrm{Aut}_\pi^{\circ,(1,1)}(X)$, where $\mathrm{Aut}_\pi^{\circ,(1,1)}(X)$ is a finite index subgroup of $\mathrm{Aut}_\pi(X)$ defined in Lemma 5.1. Then the* current-valued pairing $\eta_B(T,[\alpha])$ *has a continuous potential for any closed smooth real $(1,1)$-form α on X such that $\int_{X_s} \alpha = 0$ for $s \neq 0$.*

Proof. We first show that $u \in L^\infty(B^\circ)$ following Tate's argument in [Tat83] (see also [FT23, Remark 3.2.14]).

From Equation (5.2), for any integer $i \geq 0$, we have

$$T_*^i \pi^* u = \pi^* u = T_*^i (f + T_* \varphi - \varphi) = T_*^i f + (T_*^{i+1} - T_*)\varphi.$$

Then for $k \geq 1$ the Birkhoff sum $S_k(T, f) = \sum_{i=0}^{k-1} T_*^i f$ satisfies

$$S_k(T, f) = k\pi^* u + \varphi - T_*^k \varphi.$$

Thus, for $s \in B^\circ$ and every $k \geq 1$, we have

$$|u(s)| \leq \frac{1}{k} \sup_{X_s} |S_k(T, f)| + \frac{1}{k} \sup_{X_s} \left|\varphi - T_*^k \varphi\right| \leq \|f\|_{L^\infty(X)} + \frac{2}{k} \|\varphi\|_{L^\infty(X_s)}.$$

Fixing $s \in B^\circ$ and letting $k \to \infty$, we have $|u(s)| \leq \|f\|_{L^\infty(X)}$. Therefore, $u \in L^\infty(B^\circ)$.

On the other hand, from the relation between the potential u and the *Archimedean height pairing* in Equation (5.4), the Archimedean height pairing

$$\langle \alpha, (T^* - I)\omega \rangle(s) = u(s) - \int_{X_s} f \omega_s \in L^\infty(B^\circ)$$

is bounded.

By Corollary 4.6, $\langle \alpha, (T^* - I)\omega \rangle$ can be extended continuously on B because it is bounded and has no singularity. Using Barlet's result in Theorem 1.3, the function $s \mapsto \int_{X_s} f \omega_s$ is Hölder continuous on B.

In summary, $u(s) = \langle \alpha, (T^* - I)\omega \rangle(s) + \int_{X_s} f \omega_s$ is continuous on B. So $\eta_B(T, [\alpha])$ has a continuous potential. □

As a corollary, we have

Corollary 5.5. *Let ω be a Kähler metric on X. For any $T \in \mathrm{Aut}_\pi^{\circ,(1,1)}(X)$, the limit current*

$$\lim_{n \to \infty} \frac{(T^n)_* \omega}{n^2} = \frac{1}{2} \pi^* \eta_B(T, [T_* \omega - \omega]).$$

has a continuous potential.

We note that the convergence of currents $\frac{(T^n)_* \omega}{n^2} \to \frac{1}{2} \pi^* \eta_B(T, [T_* \omega - \omega])$ cannot be in $C^0_{\mathrm{loc}}(X \setminus X_0)$.

In particular, let ω be a Ricci flat Kähler metric on the K3 surface X, i.e.,

$$\omega^2 = \mathrm{vol}(\omega) \Omega \wedge \overline{\Omega},$$

where Ω is the normalized holomorphic $(2,0)$-form on X. Then $T_*^n \omega$ is also Ricci flat. Therefore, $\{[\frac{T_*^n \omega}{n^2}]\}$ provides a counterexample to Tosatti's question [Tos25, Question 1.5(b)].

Proof. By [FT23, Corollary 3.2.20], we have

$$\lim_{n \to \infty} \frac{(T^n)_* \omega}{n^2} = \frac{1}{2} \pi^* \eta_B(T, [T_* \omega - \omega]).$$

So $\lim_{n \to \infty} \frac{(T^n)_* \omega}{n^2}$ has continuous potential by Proposition 5.4.

We now prove that the convergence of currents $\frac{(T^n)_*\omega}{n^2} \to \frac{1}{2}\pi^*\eta_B(T, [T_*\omega - \omega])$ is not a $C^0_{\mathrm{loc}}(X \setminus X_0)$ convergence. We proceed by contradiction, assuming that $\frac{(T^n)_*\omega}{n^2} \to \frac{1}{2}\pi^*\eta_B(T, [T_*\omega - \omega])$ in $C^0_{\mathrm{loc}}(X \setminus X_0)$.

We let $\xi = T_*\omega - \omega$. Then by elementary computation (see also [FT23, Proposition 3.2.15]), we have

$$T^n_*\omega = \omega + n\xi + \mathrm{dd}^c(n\varphi - S_n(T,\varphi)) + \frac{n(n-1)}{2}\pi^*\eta_B(T,[\xi]),$$

where φ is the *preferred potential* of ξ and $S_n(T,\varphi) = \sum_{k=0}^{n-1} T^k_*\varphi$ is a Birkhoff sum.

Note that $\frac{1}{n^2}(\omega + n\xi + \frac{n(n-1)}{2}\pi^*\eta_B(T,[\xi]))$ converges to $\frac{1}{2}\pi^*\eta_B(T,[\xi])$ in $C^0_{\mathrm{loc}}(X \setminus X_0)$. Under our assumption for contradiction, we have

$$\frac{1}{n^2}\mathrm{dd}^c(n\varphi - S_n(T,\varphi)) \to_{n\to\infty} 0 \quad \text{in } C^0_{\mathrm{loc}}(X \setminus X_0). \tag{5.5}$$

It suffices to show that Equation (5.5) is impossible. We interpret φ, the *preferred potential* of $\xi = T_*\omega - \omega$, as follows: Since π is an elliptic fibration, each smooth fiber X_s is isomorphic to a torus $\mathbb{C}/\Lambda_s$, and T acts on X_s by translations. Let $\omega_{\mathrm{flat,s}}$ be the normalized flat metric on X_s. There exists $\rho_s \in C^\infty(X_s)$ such that

$$\omega_{\mathrm{flat,s}} - \omega_s = \mathrm{dd}^c\rho_s.$$

Applying T_* to this equation, we obtain $\omega_{\mathrm{flat,s}} - T_*\omega_s = \mathrm{dd}^c T_*\rho_s$, since $\omega_{\mathrm{flat,s}}$ is invariant under translation T. Then we have

$$\xi \mid_{X_s} = T_*\omega_s - \omega_s = -\mathrm{dd}^c(T_*\rho_s - \rho_s).$$

Thus, the *preferred potential* φ of ξ is equal to $T_*\rho_s - \rho_s$ up to a constant on X_s. By Ehresmann's theorem and solving equations on the region where the fibration is locally trivial, there exists $\rho \in C^\infty(X \setminus X_0)$ such that $\rho \mid_{X_s} = \rho_s$ on the smooth fibers X_s. Furthermore, $\varphi = T_*\rho - \rho + c(s)$, where $c(s)$ is a smooth function on B°.

Substituting φ by $T_*\rho - \rho + c(s)$ in Equation (5.5), we have

$$\frac{1}{n^2}\mathrm{dd}^c(n(T_*\rho - \rho) - (T^n_*\rho - \rho)) \to_{n\to\infty} 0 \quad \text{in } C^0_{\mathrm{loc}}(X \setminus X_0).$$

By smoothness of ρ on $X \setminus X_0$, the above convergence is equivalent to

$$\frac{1}{n^2}\mathrm{dd}^c(T^n_*\rho) \to_{n\to\infty} 0 \quad \text{in } C^0_{\mathrm{loc}}(X \setminus X_0). \tag{5.6}$$

We will show that Equation (5.6) is impossible.

On a small open set $U \subset B^\circ$ not containing any singular points, we can trivialize the fibration π as $U \times \mathbb{C}/\Lambda \to U$. Moreover, we can lift the $U \times \mathbb{C}/\Lambda$ to its universal covering $U \times \mathbb{C}$, and the fiber-preserving automorphism T can also be lifted, yielding the following commutative diagram:

$$\begin{tikzcd} U_s \times \mathbb{C}_z \arrow[loop left, "\widetilde{T}"] \arrow[r, "p"] \arrow[dr, "pr_1"'] & \pi^{-1}(U_s) \arrow[loop right, "T"] \arrow[d, "\pi"] \\ & U_s \end{tikzcd}$$

We can then express the action of $\widetilde{T}$ on the covering space $U_s \times \mathbb{C}_z$ as $\widetilde{T}(s,z) = (s, z + T(s))$, where $T\colon U_s \to \mathbb{C}$ is a holomorphic function. Following the argument in [Can01, Proposition 2.2] (see also [Fil22, Proposition 6.2.7], [CD23a, Lemma 3.7]), the function T is non-constant.

Lifting the convergence in Equation (5.6) to the covering space, we find it is equivalent to

$$\frac{1}{n^2}\mathrm{dd}^c(\widetilde{T}^n_*\widetilde{\rho}) \to_{n\to\infty} 0 \quad \text{in } C^0_{\mathrm{loc}}(U_s \times \mathbb{C}_z), \tag{5.7}$$

for all small open neighborhoods of points in B° and its corresponding covering, where $\widetilde{\rho}$ is the lift of ρ to the covering.

Note that $\widetilde{T}^n_*\widetilde{\rho} = \widetilde{\rho}(s, z - nT(s))$. By the holomorphic chain rule, we have

$$\begin{aligned}
\partial_s\overline{\partial}_s(\widetilde{T}^n_*\widetilde{\rho})(s,z) &= \partial_s\overline{\partial}_s(\widetilde{\rho}(s, z - nT(s)))\\
&= \partial_s((\overline{\partial}_s\widetilde{\rho})(s,z-nT(s)) - n\overline{\partial_s T}(\overline{\partial}_z\widetilde{\rho})(s,z-nT(s))) \quad (\overline{\partial}_s T(s) = 0)\\
&= (\partial_s\overline{\partial}_s\widetilde{\rho})(s,z-nT(s)) - n(\partial_s T(s))(\partial_z\overline{\partial}_s\widetilde{\rho})(s,z-nT(s))\\
&\quad - n\overline{\partial_s T}(\partial_s\overline{\partial}_z\widetilde{\rho})(s,z-nT(s)) + n^2\left|\partial_s T\right|^2(\partial_z\overline{\partial}_z\widetilde{\rho})(s,z-nT(s))
\end{aligned}$$

Then Equation (5.7) implies that

$$\left|\partial_s T\right|^2(\partial_z\overline{\partial}_z\widetilde{\rho})(s,z-nT(s)) \to_{n\to\infty} 0 \quad \text{in } C^0_{\mathrm{loc}}(U_s\times\mathbb{C}_z)$$

for all small open sets in B°.

Since T is non-constant and holomorphic, we have $\left|\partial_s T\right|^2 > 0$ except for at most finitely many $s \in U_s$. Therefore, there exists a finite set $\Sigma \subset U_s$ such that

$$(\partial_z\overline{\partial}_z\widetilde{\rho})(s,z-nT(s)) \to_{n\to\infty} 0 \quad \text{in } C^0_{\mathrm{loc}}((U_s\setminus\Sigma)\times\mathbb{C}_z).$$

Since $\widetilde{\rho}$ is periodic in z, it implies that $\partial_z\overline{\partial}_z\widetilde{\rho}(s,z) = 0$ for $s \in U_s \setminus \Sigma$. By smoothness of $\widetilde{\rho}$, we have $\partial_z\overline{\partial}_z\widetilde{\rho}(s,z) \equiv 0$ for $s \in S^\circ$. Consequently, $\mathrm{dd}^c\rho_s = 0$ on X_s for $s \in S^\circ$.

By definition of ρ, we have

$$\omega_{\mathrm{flat,s}} - \omega_s = \mathrm{dd}^c\rho_s = 0.$$

Thus, $\omega_{\mathrm{flat,s}} = \omega_s$ for $s \in S^\circ$.

To get a contradiction, we compute the Gauss curvature of $\omega_{\mathrm{flat,s}}$ and ω on smooth fibers X_s, and study them when X_s is approaching a singular fiber.

By definition, the Gauss curvature $K_{\omega_{\mathrm{flat,s}}} \equiv 0$. On the other hand, by [DY25, Lemma 10.1], the minimum of K_{ω_s} goes to $-\infty$ as $s \to 0$. This leads to a contradiction, concluding the proof that the convergence is not in $C^0_{\mathrm{loc}}(X \setminus X_0)$.

□

Appendix A. Asymptotics of a fiber integral

We prove a lemma on the asymptotics of the fiber integral of the log-norm of the canonical section of a divisor. This result also appears in other works (e.g., [Yos06, Section 4]); however, we provide a proof here in our setting for convenience and completeness.

Lemma A.1. *Let $\pi\colon X \to \mathbb{D}$ be a degeneration of algebraic curves. Assume X is smooth and we allow a general singular fiber X_0 here. Let C be one of the irreducible components of X_0 and $L = \mathcal{O}(C)$ the corresponding line bundle with Hermitian metric h.*

Let σ be the canonical section of L which vanishes exactly on C. Then, for any smooth $(1,1)$-form α on X, there exists a constant $c \in \mathbb{Q}_{>0}$ such that

$$\int_{X_s} \log \|\sigma\|_h^2 \,\alpha - \left(\int_C \alpha\right) \log |s|^2$$

extends Hölder continuously through the origin.

Proof. We consider a semi-stable reduction for the family $\pi\colon (X, X_0) \to (\mathbb{D}, \{0\})$ given by the following commutative diagram:

$$\begin{tikzcd} Y \arrow[r, "\mu"] \arrow[dr, "p"'] & X \times_{\mathbb{D}} \mathbb{D}_t \arrow[r, "pr_1"] \arrow[d, "pr_2"] & X \arrow[d, "\pi"] \\ & \mathbb{D}_t \arrow[r, "f_d"] & \mathbb{D}_s \end{tikzcd}$$

Here, $f_d(t) = t^d = s$ is a base change of degree d, and μ is a resolution of singularities such that Y is a smooth complex surface. The singular fiber $Y_0 = p^{-1}(X_0)$ is a reduced divisor with simple normal crossings. Let $\widetilde{\mu} = pr_1 \circ \mu$.

Let m be the multiplicity of C in X_0. Since $\widetilde{\mu}^* X_0 = dY_0$ and Y_0 is reduced, the strict transform $\widetilde{C}$ of C appears in Y_0 with multiplicity 1. Thus, the pullback of C as a Cartier divisor is

$$\widetilde{\mu}^* C = \frac{d}{m}\widetilde{C} + \sum_k \nu_k E_k$$

where E_k are exceptional divisors.

This induces a decomposition of the log-norm on Y:

$$\log \|\widetilde{\mu}^*\sigma\|^2_{\widetilde{\mu}^* h} = \frac{d}{m} \log \|\sigma_{\widetilde{C}}\|^2_{h_{\widetilde{C}}} + \sum_k \nu_k \log \|\sigma_{E_k}\|^2_{h_{E_k}} + u \tag{A.1}$$

where σ_D denotes the canonical section of a component $D \subset Y_0$, and u is a smooth function on Y.

For $t \neq 0$, the map $\widetilde{\mu}_t\colon Y_t \xrightarrow{\sim} X_s$ is an isomorphism. Pulling the integral back to the semi-stable model Y_t, we obtain:

$$\int_{X_s} \log \|\sigma\|_h^2 \,\alpha = \int_{Y_t} \log \|\widetilde{\mu}^*\sigma\|^2_{\widetilde{\mu}^* h} \,\widetilde{\mu}^*\alpha$$

Substituting our decomposition Equation (A.1), this splits into:

$$\frac{d}{m} \int_{Y_t} \log \|\sigma_{\widetilde{C}}\|^2_{h_{\widetilde{C}}} \,\widetilde{\mu}^*\alpha + \sum_k \nu_k \int_{Y_t} \log \|\sigma_{E_k}\|^2_{h_{E_k}} \,\widetilde{\mu}^*\alpha + \int_{Y_t} u\widetilde{\mu}^*\alpha = \int_{X_s} \log \|\sigma\|_h^2 \,\alpha \tag{A.2}$$

Since Y_0 has strictly simple normal crossings, we apply Lemma A.2 to the first two terms of Equation (A.2), which yield:

$$\begin{aligned}\int_{Y_t} \log \|\sigma_{\widetilde{C}}\|^2_{h_{\widetilde{C}}} \widetilde{\mu}^*\alpha &= \left(\int_{\widetilde{C}} \widetilde{\mu}^*\alpha\right) \log |t|^2 + R_1(t) \\ &= m\left(\int_C \alpha\right) \log |t|^2 + R_1(t) \\ \int_{Y_t} \log \|\sigma_{E_k}\|^2_{h_{E_k}} \widetilde{\mu}^*\alpha &= \left(\int_{E_k} \widetilde{\mu}^*\alpha\right) \log |t|^2 + R_{2,k}(t), \\ &= \left(\int_{\widetilde{\mu}_*[E_k]} \alpha\right) \log |t|^2 + R_{2,k}(t) \\ &= R_{2,k}(t),\end{aligned}$$

where $R_1(t), R_{2,k}(t)$ are Hölder continuous around 0.

Since u is smooth, by Barlet's result (Theorem 1.3), the third term (of Equation (A.2)) $\int_{Y_t} u\widetilde{\mu}^*\alpha =: R_3(t)$, is Hölder continuous around 0. Substituting the previous computations into Equation (A.2), we have

$$\int_{X_s} \log \|\sigma\|^2_h \alpha = d\left(\int_C \alpha\right) \log |t|^2 + \underbrace{\frac{d}{m}R_1(t) + \sum_k R_{2,k}(t) + R_3(t)}_{\text{Hölder continuous}},$$

Finally, we translate back to the original base parameter s. Since $s = t^d$, we have $\log |t|^2 = \frac{1}{d}\log |s|^2$. Therefore,

$$\int_{X_s} \log \|\sigma\|^2_h \alpha = \left(\int_C \alpha\right) \log |s|^2 + R(s),$$

and $R(s)$ is Hölder continuous.

We conclude that:

$$\int_{X_s} \log \|\sigma\|^2_h \alpha - \left(\int_C \alpha\right) \log |s|^2$$

extends Hölder continuously through the origin. □

Lemma A.2. *Let $\pi\colon \mathcal{Y} \to \mathbb{D}$ be a proper holomorphic map from a smooth complex surface to the unit disk. Assume the singular fiber $Y_0 = \pi^{-1}(0)$ is a reduced divisor with strictly simple normal crossings. Let D be an irreducible component of Y_0, σ_D be its canonical section, and h be a smooth Hermitian metric on $\mathcal{O}(D)$. For any smooth $(1,1)$-form ω on $\mathcal{Y}$, the function*

$$t \mapsto \int_{Y_t} \log \|\sigma_D\|^2_h \omega - \left(\int_D \omega\right) \log |t|^2$$

extends Hölder continuously to $t = 0$.

Proof. Fix a finite open cover $\{U_j\}$ of Y_0 in $\mathcal{Y}$ by coordinate polydisks $U_j \cong \mathbb{D}^2$ such that on each U_j, π is given either by $t = z_1$ (smooth points of Y_0) or $t = z_1 z_2$ (nodes of Y_0). Let $\{\rho_j\}$ be a smooth partition of unity subordinate to $\{U_j\}$.

On each U_j, we decompose the metric as $\log \|\sigma_D\|_h^2 = \log |f_D|^2 + u_j$, where $f_D \in \mathcal{O}(U_j)$ defines $D \cap U_j$ and u_j is a smooth function. By Barlet's theorem on fiber integral (Theorem 1.3), $t \mapsto \int_{Y_t \cap U_j} \rho_j u_j \omega$ extends Hölder continuously to $t = 0$. We analyze the singular integral $I_j(t) = \int_{Y_t \cap U_j} \rho_j \log |f_D|^2 \omega$ case by case:

Case 1: $U_j \cap D = \emptyset$. Here, f_D is a non-vanishing holomorphic function on U_j. Thus $\log |f_D|^2$ is smooth, and $I_j(t)$ is smooth around $t = 0$. In this case,

$$I_j(t) = \left(\int_{D \cap U_j} \rho_j \omega \right) \log |t|^2 + I_j(t).$$

Case 2: U_j intersects D outside the nodes. Here, $\pi = z_1$ and $f_D = z_1$. Thus $I_j(t) = \log |t|^2 \int_{Y_t \cap U_j} \rho_j \omega$. Since π on U_j is a submersion, by Ehresmann's theorem and Taylor expansion, we have $\int_{Y_t \cap U_j} \rho_j \omega = \int_{\{z_1 = t\} \cap U_j} \rho_j \omega = \int_{\{z_1 = 0\} \cap U_j} \rho_j \omega + O(|t|) = \int_{D \cap U_j} \rho_j \omega + O(|t|)$. Multiplying by $\log |t|^2$, we obtain

$$I_j(t) = \left(\int_{D \cap U_j} \rho_j \omega \right) \log |t|^2 + R_j(t),$$

where $R_j(t) = O(|t| \log |t|)$ is Hölder continuous around $t = 0$.

Case 3: U_j contains a node $D \cap D'$. Here, the map is given by $\pi = z_1 z_2$, with D defined by $z_1 = 0$ and D' defined by $z_2 = 0$. Thus $f_D = z_1$. The fiber $Y_t \cap U_j$ is the restricted annulus $\{(z_1, z_2) \in U_j \mid z_1 z_2 = t\}$.

Let $\eta = \rho_j \omega$. Since η is a smooth form with compact support in $U_j \cong \mathbb{D}^2$, we can write it globally on the chart as $\eta = \frac{i}{2} \sum_{k,l=1}^2 \eta_{k\bar{l}} dz_k \wedge d\bar{z}_l$. We want to evaluate the asymptotic behavior of

$$I_j(t) = \int_{Y_t \cap U_j} \log |z_1|^2 \eta$$

We partition the domain of integration on the annulus $Y_t \cap U_j$ into two regions: $V_1(t) = \{|t|^{1/2} \le |z_1| < 1\}$ and $V_2(t) = \{|t|^{1/2} < |z_2| < 1\}$.

On $V_1(t)$, we use z_1 as the coordinate. Substituting $z_2 = t/z_1$ and $dz_2 = -(t/z_1^2) dz_1$, the restriction of η is:

$$\eta|_{Y_t} = \left(\eta_{1\bar{1}} - \eta_{1\bar{2}} \frac{\bar{t}}{\bar{z}_1^2} - \eta_{2\bar{1}} \frac{t}{z_1^2} + \eta_{2\bar{2}} \frac{|t|^2}{|z_1|^4} \right) \frac{i}{2} dz_1 \wedge d\bar{z}_1$$

where the smooth coefficients $\eta_{k\bar{l}}$ are evaluated at $(z_1, t/z_1)$. By the smoothness of η, we can Taylor expand the leading coefficient as $\eta_{1\bar{1}}(z_1, t/z_1) = \eta_{1\bar{1}}(z_1, 0) + O(|t| / |z_1|)$.

Integrating the cross-term and higher-order errors against $\log |z_1|^2$ over $V_1(t)$ yields bounded remainders. For example, passing to polar coordinates $r = |z_1|$:

$$\int_{|t|^{1/2}}^1 |\log r| \cdot O\left(\frac{|t|}{r^2} \right) r \, dr = O\left(|t| \int_{|t|^{1/2}}^1 \frac{|\log r|}{r} dr \right) = O(|t| \log^2 |t|)$$

Similar bounds hold for the $O(|t|^2 / r^4)$ and $O(|t| / r)$ terms. Thus, the integral over $V_1(t)$ is tightly dominated by the leading term:

$$\int_{V_1(t)} \log |z_1|^2 \eta|_{Y_t} = \int_{|t|^{1/2} \le |z_1| < 1} \log |z_1|^2 \eta_{1\bar{1}}(z_1, 0) \frac{i}{2} dz_1 \wedge d\bar{z}_1 + O(|t| \log^2 |t|)$$

Extending the domain of this integral to the full local component $D' \cap U_j$ (where $z_2 = 0$) introduces an error bounded by the integral over the missing inner disk $\{|z_1| < |t|^{1/2}\}$. This hole has a volume integral of $\int_0^{|t|^{1/2}} |\log r|\, r\, dr = O(|t| \log |t|)$. Hence:

$$I_{j,1}(t) = \underbrace{\int_{D' \cap U_j} \log |z_1|^2 \, \eta|_{D'}}_{\text{constant}} + O(|t| \log^2 |t|).$$

On $V_2(t)$, we switch to the coordinate z_2. Here, the log-norm splits exactly as $\log |z_1|^2 = \log |t|^2 - \log |z_2|^2$.

Similarly, the restriction of η in the z_2 coordinate is:

$$\eta|_{Y_t} = \left(\eta_{2\bar{2}} - \eta_{1\bar{2}} \frac{t}{z_2^2} - \eta_{2\bar{1}} \frac{\bar{t}}{\bar{z}_2^2} + \eta_{1\bar{1}} \frac{|t|^2}{|z_2|^4} \right) \frac{i}{2} dz_2 \wedge d\bar{z}_2$$

with coefficients evaluated at $(t/z_2, z_2)$.

The term involving $-\log |z_2|^2$ behaves exactly like the $V_1(t)$ integral, yielding a constant relative to t minus a remainder:

$$-\int_{V_2(t)} \log |z_2|^2 \, \eta|_{Y_t} = \underbrace{-\int_{D \cap U_j} \log |z_2|^2 \, \eta|_D}_{\text{constant}} + O(|t| \log^2 |t|)$$

For the $\log |t|^2$ term, we must integrate the volume-like form $\eta|_{Y_t}$ over $V_2(t)$. Applying the exact same Taylor expansion and error bounds, the dominant term is $\eta_{2\bar{2}}(0, z_2)$. We have:

$$\int_{V_2(t)} \eta|_{Y_t} = \int_{|t|^{1/2} \le |z_2| < 1} \eta_{2\bar{2}}(0, z_2) \frac{i}{2} dz_2 \wedge d\bar{z}_2 + \int_{|t|^{1/2}}^1 O\left(\frac{|t|}{r^2} \right) r\, dr$$

The error integral evaluates to $O(|t|\, |\log |t||)$. The difference between the main integral and the full integral over $D \cap U_j$ is the missing disk $\{|z_2| < |t|^{1/2}\}$, which adds a simple $O(|t|)$ error. Thus:

$$\int_{V_2(t)} \eta|_{Y_t} = \int_{D \cap U_j} \eta|_D + O(|t| \log |t|)$$

Multiplying this volume result by $\log |t|^2$ gives $\left(\int_{D \cap U_j} \rho_j \omega \right) \log |t|^2 + O(|t| \log^2 |t|)$.

Combining both regions for the entire chart U_j, we have

$$I_j(t) = \left(\int_{D \cap U_j} \rho_j \omega \right) \log |t|^2 + \widetilde{R}_j(t),$$

where $\widetilde{R}_j(t) = \text{constant} + O(|t| \log^2 |t|)$ is Hölder continuous for any exponent $0 < \alpha < 1$ around $t = 0$. $\square$

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Courant Institute of Mathematical Sciences, New York University, 251 Mercer St, New York, NY 10012

Email address, `junyu.cao@nyu.edu`

Homepage, `https://junyucao1024.github.io`.