ASYMPTOTICS OF SMALL EIGENVALUES ON DEGENERATIONS OF KÄHLER MANIFOLDS

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Abstract. We derive the exact asymptotic rates of the small eigenvalues of the Laplacian on one-parameter degenerations of compact Kähler manifolds equipped with induced background metrics. This generalizes a recent result of Dai and Yoshikawa to higher dimensions. To achieve this, we combine Li's uniform Skoda inequality with the method of auxiliary Monge-Ampère equations, introduced by Guo–Phong–Song–Sturm–Tong and adapted by Guedj–Tô. As an application, we establish estimates for degenerations of compact Kähler manifolds with reducible singular fibers.

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Introduction

Let $\pi\colon X \to S \simeq \mathbb{D}$ be a proper surjective holomorphic map from a complex manifold X to a Riemann surface biholomorphic to the unit disk. We assume that π has connected fibers of complex dimension n, that $X_0 := \pi^{-1}(0)$ is the unique singular

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fiber, and that X is equipped with a Kähler metric ω_X. We refer to this datum as a one-parameter degeneration of compact Kähler manifolds, or Kähler degeneration for short.

In this paper, we study the asymptotics of the small eigenvalues of the Laplacian on the smooth fibers $(\pi^{-1}(s) =: X_s, \omega_X|_{X_s} =: \omega_s)$ for $s \neq 0$ near the singular fiber.

In [Yos97], Yoshikawa proved that the spectrum of the Laplacian on (X_s, ω_s) varies continuously and converges, after a suitable base change and normalization, to the spectrum of the limiting central fiber as $s \to 0$. Precisely, write $X_0 = \sum_\alpha m_\alpha D_\alpha$ as the sum of its irreducible components $(D_\alpha)_{1 \leq \alpha \leq a}$, and let $m = \prod_\alpha m_\alpha$. Consider the following commutative diagram,

$$\begin{array}{ccccc} \widehat{F^{-1}X} & \xrightarrow{\iota} & F^{-1}X := X \times_{\mathbb{D}_s} \mathbb{D}_t & \xrightarrow{F} & X \\ & & \downarrow \Pi & & \downarrow \pi \\ & \searrow^{\widehat{\Pi}} & \mathbb{D}_t & \xrightarrow{t \mapsto t^m} & \mathbb{D}_s \end{array}$$

where $F^{-1}X$ is the base-change of the degeneration and $\iota \colon \widehat{F^{-1}X} \to F^{-1}X$ is the normalization of $F^{-1}X$. We define $Z = \widehat{\Pi}^{-1}(0)$, which is a reduced divisor. Then Yoshikawa's continuity theorem takes the following form.

Theorem 0.1 ([Yos97, Main theorem])**.** *As $s \to 0$, the spectrum of the Laplacian of (X_s, ω_s) converges to the spectrum of the Laplacian of (Z_{reg}, g_Z), where g_Z is the restriction of the background metric $\iota^* F^* \omega_X$. We remark that $F \circ \iota$ is an immersion on the smooth part of Z, so $\iota^* F^* \omega_X$ is a well-defined metric on Z_{reg}.*

Although [Yos97] is stated for projective degenerations, the same argument applies to Kähler degenerations once one uses a uniform Sobolev inequality for Kähler families; see [Tos10, Lemma 3.2].

Let $N_Z := \#\mathrm{Irr}(Z)$ be the number of irreducible components of Z. This is the spectral count that controls the small eigenvalues. Because Z_{reg} has N_Z connected components of finite volume, there are exactly N_Z zero eigenvalues in the spectrum of the Laplacian of (Z_{reg}, g_Z). On the other hand, for $s \neq 0$, the fiber X_s is connected, so the Laplacian of (X_s, ω_s) has exactly one zero eigenvalue, denoted by $\lambda_0(s) = 0$. Let

$$0 = \lambda_0(s) < \lambda_1(s) \leq \lambda_2(s) \leq \cdots \leq \lambda_{N_Z}(s)$$

be the first $N_Z + 1$ eigenvalues of the Laplacian of (X_s, ω_s). By Yoshikawa's continuity theorem (Theorem 0.1), exactly N_Z eigenvalues of X_s must converge to 0 as $s \to 0$. Therefore, we have

$$\lambda_1(s), \cdots, \lambda_{N_Z - 1}(s) \to 0, \quad \lambda_{N_Z}(s) \to \lambda_{+,Z} \quad \text{as } s \to 0,$$

where $\lambda_{+,Z} > 0$ is the first strictly positive eigenvalue of (Z_{reg}, g_Z).

We call $\lambda_1(s), \cdots, \lambda_{N_Z - 1}(s)$ the **small eigenvalues** of the Laplacian on (X_s, ω_s). Thus small eigenvalues exist if and only if $N_Z \geq 2$. When X_0 is reduced, we have $Z = X_0$, so N_Z is the number of irreducible components of X_0. In particular, if $\mathrm{Supp}(X_0)$ is reducible, then small eigenvalues occur.

Our first main result gives a logarithmic lower bound for the first positive eigenvalue.

Theorem 0.2 (=Theorem 1.13). *Let $\lambda_1(s) > 0$ denote the first eigenvalue of the Laplacian on (X_s, ω_s) for $s \neq 0$. Then, there exists a uniform constant $C > 0$, independent of s, such that*

$$C \left|\log^{-1} |s|\right| \leq \lambda_1(s)$$

for $0 < |s| \ll 1$.

When $N_Z \geq 2$, so that $\lambda_1(s)$ is a small eigenvalue, this $\left|\log^{-1} |s|\right|$ lower bound is optimal.

Theorem 0.3 (=Theorem 2.3). *Suppose that $N_Z \geq 2$. Then, there exists a uniform constant $C' > 0$ such that*

$$\lambda_1(s) \leq \cdots \leq \lambda_{N_Z - 1}(s) \leq C' \left|\log^{-1} |s|\right|$$

for $0 < |s| \ll 1$.

Combining Theorem 0.2 and Theorem 0.3 yields a complete characterization of the asymptotic behavior of all small eigenvalues.

Theorem 0.4 (Main theorem). *Suppose a small eigenvalue exists, i.e., $N_Z \geq 2$. Then there exist constants $C_1, C_2 > 0$ such that for all $0 < |s| \leq \frac{1}{2}$,*

$$C_1 \left|\log^{-1} |s|\right| \leq \lambda_1(s) \leq \cdots \leq \lambda_{N_Z - 1}(s) \leq C_2 \left|\log^{-1} |s|\right|.$$

Remark 0.5. By continuity and strict positivity of the eigenvalues away from the singular fiber, the domain of validity can be naturally extended from $0 < |s| \ll 1$ to $0 < |s| \leq \frac{1}{2}$.

Remark 0.6. The main theorem generalizes the central result of Dai and Yoshikawa [DY25, Theorem 0.2] for degenerations of Riemann surfaces, where the singular fiber X_0 is assumed to be reduced. In contrast, Theorem 0.4 allows non-reduced and more general singular fibers, thereby resolving a question raised in [DY25, Problem 9.1].

Remark 0.7. In [Gro92], Gromov established a polynomial spectral gap, $\lambda_1(s) \geq c\,|s|^{\alpha}$, for degenerations of semi-algebraic submanifolds of the standard sphere $\mathbb{S}^{N-1}$. Our logarithmic rate $\left|\log^{-1} |s|\right|$ is sharper than such a polynomial bound. Moreover, Theorem 0.4 shows that this logarithmic rate is optimal whenever small eigenvalues exist.

We briefly sketch the proof of Theorem 0.4. For the upper bound in Theorem 0.3, we adapt the test-function construction of Dai and Yoshikawa [DY25, Section 6] and extend it to higher dimensions in Section 2. The key point is that we carry out the relevant estimates directly on a semistable model of the degeneration, which allows us to remove the reducedness assumption imposed in [DY25].

The lower bound in Theorem 0.2 is subtler. As observed in [DY25, Appendix], the curvature of (X_s, ω_s) blows up as $s \to 0$, so classical Riemannian eigenvalue estimates based on uniform lower curvature bounds are not available in our setting. We therefore take a purely complex-analytic route. More precisely, our method combines Li's uniform Skoda inequality for plurisubharmonic functions [Li24] with the method of auxiliary Monge–Ampère equations for estimating Green's functions [GPS24, GT25].

By contrast, Dai and Yoshikawa used analytic torsion [BB90] to bypass the curvature blow-up, but that argument is specific to degenerations of Riemann surfaces. This is why a new complex-analytic approach is needed in higher dimensions.

This article is organized as follows. In Section 1 and Section 2, we establish the lower and upper bounds following the strategy outlined above. In Section 3.A, we derive estimates for degenerating families of plurisubharmonic functions and for solutions to Poisson equations, valid for general singular fibers. In Section 3.B, we present examples of the small eigenvalue phenomenon for degenerating families equipped with Kähler–Einstein metrics. Finally, in Section 3.C, we propose a conjectural non-Archimedean interpretation of this behavior.

Notation and conventions

On a complex manifold X, we define $\mathrm{d}^c = \frac{\sqrt{-1}}{2}(\overline{\partial} - \partial)$, so $\mathrm{dd}^c = \sqrt{-1}\partial\overline{\partial}$.

On a Kähler manifold (X^n, ω) with $\omega = \sqrt{-1}\sum g_{j\bar{k}} dz_j \wedge d\bar{z}_k$, we use the positive definite analyst's Laplacian $\Delta_\omega f = -\sum g^{j\bar{k}} \frac{\partial^2 f}{\partial z_j \partial \bar{z}_k}$. Under our normalization of dd^c, we have

$$-\Delta_\omega f = \mathrm{tr}_\omega(\mathrm{dd}^c f) = \frac{n\mathrm{dd}^c f \wedge \omega^{n-1}}{\omega^n},$$
$$|\mathrm{d}f|^2_\omega = \mathrm{tr}_\omega(\mathrm{d}f \wedge \mathrm{d}^c f) = \frac{n\mathrm{d}f \wedge \mathrm{d}^c f \wedge \omega^{n-1}}{\omega^n},$$

for every smooth function f.

For a closed smooth differential form α on a compact differential manifold, we use $[\alpha]$ to denote its de Rham cohomology class.

We use $\mathbb{D} := \{z \in \mathbb{C} \colon \ |z| < 1\}$ to denote the unit disk in the complex plane, and $\mathbb{D}^\circ := \mathbb{D} \setminus \{0\}$ the punctured unit disk. For a positive number $r > 0$, we also define $\mathbb{D}_r := \{z \in \mathbb{C} \colon \ |z| < r\}$ and $\mathbb{D}_r^\circ := \mathbb{D}_r \setminus \{0\}$.

We use the following asymptotic analysis notation. Let (X, d) be a metric space, fix a point $s_0 \in X$, and let f and g be functions taking values in a partially ordered $\mathbb{R}$-vector space. Assuming f and g are defined on a punctured neighborhood of s_0, we say that:

- $f = O(g)$ (or $f \lesssim g$) as $s \to s_0$ if there exists a constant $C > 0$, independent of s, such that $f \le Cg$ on a sufficiently small punctured neighborhood of s_0.
- $f = \Theta(g)$ (or $f \asymp g$) as $s \to s_0$ if both $f = O(g)$ and $g = O(f)$. Equivalently, this holds if there exist constants $0 < C_1 \le C_2$ such that $C_1 g \le f \le C_2 g$ on a sufficiently small punctured neighborhood of s_0.

Throughout this paper, we use the fact that every 1-parameter Kähler degeneration admits a semistable reduction after a proper base change; we refer to [KKMSD73] for this result. Note that any semistable reduction of a Kähler degeneration is a Kähler manifold.

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1. Lower bound

In this section, we prove the lower bound of small eigenvalues in Theorem 0.2. We prove it using rescaled uniform Skoda estimates in Section 1.A together with the method of auxiliary Monge-Ampère equations in Section 1.B. Finally, the lower bound is proved in Section 1.C.

1.A. **A rescaled uniform Skoda estimate**

In this section, we prove the following rescaled uniform Skoda inequality.

Proposition 1.1. *Let $\pi\colon (X,\omega_X) \to \mathbb{D}_s$ be a degeneration of Kähler manifolds of complex dimension n. Assume X_0 is the unique singular fiber (we allow X_0 to be a general singular fiber).*

Let β be a closed semi-positive form on X. Assume that the restriction $\beta_s = \beta|_{X_s}$ is Kähler for $s \neq 0$, and denote its volume by $V_{\beta_s} = \int_{X_s} \beta_s^n$. Let $\mathrm{d}V_{\beta_s} = \beta_s^n / V_{\beta_s}$ be the normalized volume form on X_s for $s \neq 0$.

Consider the rescaled relative Kähler form $\widetilde{\beta}_s = \frac{\beta_s}{|\log|s||}$ on $X \setminus X_0$. Then the following Skoda-type estimate holds: there exist constants $\alpha, C > 0$ independent of s, such that for all $0 < |s| \ll 1$ and for any $\varphi \in \mathrm{PSH}(X_s, \widetilde{\beta}_s)$ with $\sup_{X_s} \varphi = 0$, we have

$$\int_{X_s} \exp(-\alpha\varphi)\mathrm{d}V_{\beta_s} \leq C.$$

Remark 1.2. In [Li24, Theorem 1.3], Li proved a similar rescaled Skoda estimate for normalized Calabi-Yau measures. Although the volume forms $\mathrm{d}V_{\beta_s}$ differ significantly from the Calabi-Yau measures near the singular fiber, the underlying proof strategies are analogous.

Before proceeding with the proof of Proposition 1.1, we establish the following reductions.

- **Reduction to the semistable case:** We can assume that the degeneration $\pi\colon X \to \mathbb{D}_s$ is semistable (i.e., the central fiber X_0 is a reduced, simple normal crossing divisor).

  Proof of Reduction. By performing a semistable reduction after a suitable base change of $\pi\colon X \to \mathbb{D}_s$, we obtain the following commutative diagram:

  $$\begin{array}{ccc} Y & \xrightarrow{\ \mu\ } & X \\ \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle \pi} \\ \mathbb{D}_t & \xrightarrow{\ f_d\ } & \mathbb{D}_s \end{array}, \quad f_d(t) = t^d.$$

  Here, the central fiber Y_0 is a reduced snc divisor in Y. Let $\beta' = \mu^*\beta$. The pullback β' is a closed semi-positive form on Y and restricts to a relative Kähler form on $Y \setminus Y_0 \to \mathbb{D}_t^\circ$.

Given any $\varphi \in \mathrm{PSH}(X_s, \widetilde{\beta}_s)$ with $\sup_{X_s} \varphi = 0$, its pullback satisfies $d(\mu^*\varphi) \in \mathrm{PSH}(Y_t, \widetilde{\beta}'_t)$, where $\widetilde{\beta}'_t = \frac{\beta'}{|\log|t||}$ is the rescaled form on $Y \setminus Y_0$. Assuming the estimate holds in the semistable setting (applied to (Y, β') and $d(\mu^*\varphi)$), there exist uniform constants $\alpha, C > 0$ such that

$$\int_{Y_t} \exp(-(\alpha d)\mu^*\varphi)\mathrm{d}V_{\beta'_t} \leq C.$$

Since the normalized volume forms satisfy $\mathrm{d}V_{\beta'_t} = \mu^*\mathrm{d}V_{\beta_s}$, it immediately follows that

$$\int_{X_s} \exp(-(\alpha d)\varphi)\mathrm{d}V_{\beta_s} = \int_{Y_t} \exp(-(\alpha d)\mu^*\varphi)\mathrm{d}V_{\beta'_t} \leq C,$$

which validates the reduction. □

- **Reduction to a globally Kähler form:** We can assume that β is a Kähler form on the total space X, rather than merely a semi-positive form.

Proof of Reduction. Suppose the result holds for any globally Kähler form γ on X. Let β be a closed semi-positive form on X. Since $\pi \colon X \to \mathbb{D}_s$ is proper and γ is strictly positive, there exists a constant $C_0 > 0$ such that $0 \leq \beta \leq C_0\gamma$ restricted to a smaller disk, say $X|_{\mathbb{D}_{1/2}}$.

Consequently, on each fiber X_s with $|s| \leq \frac{1}{2}$, the volume forms satisfy:

$$\mathrm{d}V_{\beta_s} \leq \frac{V_{\gamma_s} C_0^n}{V_{\beta_s}} \mathrm{d}V_{\gamma_s} =: C_1 \mathrm{d}V_{\gamma_s},$$

where $C_1 > 0$ depends only on C_0 and the cohomology classes of β and γ.

Furthermore, if $\varphi \in \mathrm{PSH}(X_s, \frac{\beta_s}{|\log|s||})$, then $\varphi \in \mathrm{PSH}(X_s, \frac{C_0\gamma_s}{|\log|s||})$, which implies $\frac{\varphi}{C_0} \in \mathrm{PSH}(X_s, \frac{\gamma_s}{|\log|s||})$.

Applying our assumption for the Kähler form γ to the function $\frac{\varphi}{C_0}$ (which satisfies $\sup_{X_s} \frac{\varphi}{C_0} = 0$), there exist constants $\alpha, C > 0$ independent of s such that for $0 < |s| \ll 1$,

$$\int_{X_s} \exp\left(-\alpha \frac{\varphi}{C_0}\right) \mathrm{d}V_{\gamma_s} \leq C.$$

Using the volume bound $\mathrm{d}V_{\beta_s} \leq C_1 \mathrm{d}V_{\gamma_s}$, we deduce:

$$\int_{X_s} \exp\left(-\frac{\alpha}{C_0}\varphi\right) \mathrm{d}V_{\beta_s} \leq C \cdot C_1.$$

By setting $\alpha' = \alpha/C_0$ and $C' = C \cdot C_1$, we recover the Skoda estimate for the semi-positive form β. □

Consequently, to establish Proposition 1.1, it suffices to prove the following simplified proposition.

Proposition 1.3. *Let $\pi \colon \mathcal{Y} \to \mathbb{D}_s$ be a degeneration of compact Kähler manifolds of complex dimension n. Assume the central fiber $\mathcal{Y}_0$ is the unique singular fiber and that it is a reduced simple normal crossing (snc) divisor; that is, the degeneration is semistable.*

Let γ be a Kähler form on the total space $\mathcal{Y}$, and denote its restriction to the fiber $\mathcal{Y}_s$ by $\gamma_s = \gamma|_{\mathcal{Y}_s}$. For $s \neq 0$, let $V_{\gamma_s} = \int_{\mathcal{Y}_s} \gamma_s^n$ be the volume of the fiber. Since the fibers *are cohomologous,* $V_{\gamma_s} \equiv V$ *is a uniform constant. Define the normalized volume form* $\mathrm{d}V_{\gamma_s} = \gamma_s^n / V_{\gamma_s}$.

Consider the rescaled Kähler forms $\widetilde{\gamma}_s = \frac{\gamma_s}{|\log|s||}$ *on the fibers* $\mathcal{Y}_s$ *for* $s \neq 0$. *Then the following Skoda-type estimate holds: there exist constants* $\alpha, C > 0$ *independent of* s, *such that for all* $0 < |s| \ll 1$ *and for any* $\varphi \in \mathrm{PSH}(\mathcal{Y}_s, \widetilde{\gamma}_s)$ *with* $\sup_{\mathcal{Y}_s} \varphi = 0$, *we have*

$$\int_{\mathcal{Y}_s} \exp(-\alpha\varphi) \mathrm{d}V_{\gamma_s} \leq C.$$

We prove the proposition following the local setup of [Li24, Section 2]. Let E_i $(i \in I)$ be the irreducible components of the reduced simple normal crossing divisor $\mathcal{Y}_0$, and let d_γ be the distance induced by the ambient Kähler metric γ.

For each subset $J \subset I$ such that $E_J := \bigcap_{i \in J} E_i \neq \emptyset$, Li considers the corresponding stratum of $\mathcal{Y}_s$ in [Li24, Section 2],

$$E_J^0 = \{y \in \mathcal{Y}_s \colon d_\gamma(y, E_J) \lesssim \epsilon\} \setminus \bigcup_{J' \supsetneq J} \{y \in \mathcal{Y}_s \colon d_\gamma(y, E_{J'}) \lesssim \epsilon\}.$$

Thus E_J^0 may be regarded as an ϵ-tubular neighborhood of E_J in $\mathcal{Y}_s$, with the deeper strata removed. After shrinking the base and choosing $\epsilon > 0$ sufficiently small, these regions cover $\mathcal{Y}_s$ for all $0 < |s| \ll 1$. We remark that all parameters defining E_J^0 (e.g., ϵ and constants in $\lesssim$) are independent of s.

Fix such a subset J, and write $p = |J| - 1$. After reindexing the components in J, we may assume $J = \{0, 1, \dots, p\}$. Around any point of $E_J \setminus \bigcup_{J' \supsetneq J} E_{J'}$, semistability gives an E_J-adapted coordinate chart $\{z_i\}_{i=0}^n$ such that z_j is a local defining function of E_j for $0 \leq j \leq p$, and

$$s = z_0 \cdots z_p.$$

In such a chart,

$$\gamma \asymp \sum_{i=0}^{n} \sqrt{-1}\, \mathrm{d}z_i \wedge \mathrm{d}\overline{z}_i.$$

Restricting to $\mathcal{Y}_s$, and using that $V_{\gamma_s} = V$ is independent of s, we obtain locally

$$\mathrm{d}V_{\gamma_s} \asymp \sum_{j=0}^{p} \left(\prod_{\substack{0 \leq i \leq p \\ i \neq j}} \sqrt{-1}\, \mathrm{d}z_i \wedge \mathrm{d}\overline{z}_i \right) \wedge \left(\prod_{k=p+1}^{n} \sqrt{-1}\, \mathrm{d}z_k \wedge \mathrm{d}\overline{z}_k \right).$$

Following [Li24, Lemma 2.6], we introduce log scales on E_J^0. Fix an E_J-adapted chart intersecting E_J^0 with $\mathbb{C}^*$-coordinates $z_1, \dots, z_p$ and $\mathbb{C}$-coordinates $z_{p+1}, \dots, z_n$. For a point q on this chart, we refer to the subregion

$$\{\frac{1}{2} |z_i(q)| \lesssim |z_i| \lesssim 2 |z_i(q)|,\, 1 \leq i \leq p\}$$

as a log scale.

On each log scale we use the log measure

$$\mathrm{d}\nu_{\log} = \left(\prod_{i=1}^{p} \sqrt{-1}\, \mathrm{d}\log z_i \wedge \mathrm{d}\log \overline{z}_i \right) \wedge \left(\prod_{k=p+1}^{n} \sqrt{-1}\, \mathrm{d}z_k \wedge \mathrm{d}\overline{z}_k \right).$$

We now compare $\mathrm{d}V_{\gamma_s}$ and $\mathrm{d}\nu_{\log}$. Since $s = z_0 \cdots z_p$, on $\mathcal{Y}_s$ we have

$$0 = \mathrm{d}\log s = \mathrm{d}\log z_0 + \cdots + \mathrm{d}\log z_p.$$

Hence, for every $0 \le j \le p$,

$$\prod_{i=1}^{p} \sqrt{-1}\,\mathrm{d}\log z_i \wedge \mathrm{d}\log \overline{z}_i = \prod_{\substack{0\le i\le p\\ i\neq j}} \frac{\sqrt{-1}}{|z_i|^2}\,\mathrm{d}z_i \wedge \mathrm{d}\overline{z}_i.$$

Therefore,

$$\mathrm{d}V_{\gamma_s} \asymp \sum_{j=0}^{p} \left(\prod_{\substack{0\le i\le p\\ i\neq j}} |z_i|^2 \right) \mathrm{d}\nu_{\log} = |s|^2 \left(\sum_{j=0}^{p} |z_j|^{-2} \right) \mathrm{d}\nu_{\log}. \tag{1.1}$$

With these preliminaries in place, the proposition follows by combining Li's local Skoda estimate on each log scale ([Li24, Corollary 2.8]) with a bounded-overlap covering argument in [Li24, Theorem 2.9].

Proof of Proposition 1.3. We choose the charts so that each point on $\mathcal{Y}_s$ is covered by at most K log scales, where K is a uniform constant independent of s (see [Li24, Theorem 2.9]).

By the local Skoda estimate in [Li24, Corollary 2.8], there exist uniform constants $\alpha, A > 0$ such that on each log scale, we have

$$\int_{\mathrm{loc}} \exp(-\alpha\varphi)\mathrm{d}\nu_{\log} \le A \int_{\mathrm{loc}} \mathrm{d}\nu_{\log}. \tag{1.2}$$

On each log scale, define the weight function

$$W(z) := \frac{\mathrm{d}V_{\gamma_s}}{\mathrm{d}\nu_{\log}}.$$

From Equation (1.1),

$$\mathrm{d}V_{\gamma_s} \asymp \underbrace{\sum_{j=0}^{p} \left(\prod_{\substack{i\neq j\\ 0\le i\le p}} |z_i|^2 \right)}_{\asymp W(z)} \mathrm{d}\nu_{\log},$$

we obtain $W(z) \asymp \sum_{j=0}^{p} \left(\prod_{i\neq j,\, 0\le i\le p} |z_i|^2 \right) = |s|^2 \sum_{j=0}^{p} |z_j|^{-2}$. (Here, $\asymp$ denotes uniform equivalence independent of s).

By definition of log scales, we have

$$1 \le \frac{\max_{\mathrm{loc}} |z_i|}{\min_{\mathrm{loc}} |z_i|} \lesssim 4, \quad 1 \le i \le p.$$

Since $|z_0| = \frac{|s|}{\prod_{i=1}^{p} |z_i|}$, it follows that

$$1 \le \frac{\max_{\mathrm{loc}} |z_0|}{\min_{\mathrm{loc}} |z_0|} \lesssim 4^p.$$

Thus, for the weight function on each log scale, we have

$$1 \le \frac{\max_{\mathrm{loc}} W(z)}{\min_{\mathrm{loc}} W(z)} \lesssim \frac{\sum_{j=0}^{p}(\min_{\mathrm{loc}}|z_j|)^{-2}}{\sum_{j=0}^{p}(\max_{\mathrm{loc}}|z_j|)^{-2}} \lesssim 4^{2p} \lesssim 4^{2n}. \tag{1.3}$$

(We use $\lesssim$ to indicate a uniform upper bound independent of s).

Therefore, we deduce a new local Skoda estimate with respect to the measure $\mathrm{d}V_{\gamma_s}$:

$$\begin{aligned}
\int_{\mathrm{loc}} \exp(-\alpha\varphi)\mathrm{d}V_{\gamma_s} &\le \max_{\mathrm{loc}} W(z) \int_{\mathrm{loc}} \exp(-\alpha\varphi)\mathrm{d}\nu_{\log} \\
&\le A\left(\max_{\mathrm{loc}} W(z)\right)\int_{\mathrm{loc}} \mathrm{d}\nu_{\log} \quad \text{(Using } eq.\ (1.2)\text{)} \\
&= A\left(\max_{\mathrm{loc}} W(z)\right)\int_{\mathrm{loc}} \frac{\mathrm{d}V_{\gamma_s}}{W(z)} \\
&\le A\frac{\max_{\mathrm{loc}} W(z)}{\min_{\mathrm{loc}} W(z)}\int_{\mathrm{loc}} \mathrm{d}V_{\gamma_s} \\
&\lesssim 4^{2n}A\int_{\mathrm{loc}} \mathrm{d}V_{\gamma_s} \quad \text{(Using } eq.\ (1.3)\text{)} \\
&\le A'\int_{\mathrm{loc}} \mathrm{d}V_{\gamma_s},
\end{aligned}$$

where A' is another uniform constant.

Summing over the local Skoda estimates from all log scales, $\int_{\mathcal{Y}_s} \exp(-\alpha\varphi)\mathrm{d}V_{\gamma_s}$ is bounded by

$$A'\sum_{\text{log scale}}\int_{\mathrm{loc}} \mathrm{d}V_{\gamma_s} \le A'K\int_{\mathcal{Y}_s} \mathrm{d}V_{\gamma_s} = A'K =: C.$$

□

Remark 1.4. Although [Li24, Section 2] is written in the projective Calabi–Yau setting, we only use the local part of Li's argument, namely the log-scale L^1 estimate and the resulting local Skoda estimate. These arguments are carried out in semistable coordinates $s = z_0 \cdots z_p$, after adding a bounded local potential for the rescaled background form so that the functions become genuinely plurisubharmonic. They use the uniform equivalence of the ambient Kähler metric with the Euclidean metric in these charts, compactness of the central fiber to choose finitely many charts and uniformly bounded log-scale covers, and integration by parts on the smooth compact fibers. No projective embedding, polarization, or Calabi–Yau volume form is used in these local estimates. Therefore [Li24, Lemma 2.6 and Corollary 2.8] apply verbatim in the present semistable compact Kähler setting. The passage from Li's log measure $\mathrm{d}\nu_{\log}$ to $\mathrm{d}V_{\gamma_s}$ is exactly the bounded-ratio comparison of the weight $W = \mathrm{d}V_{\gamma_s}/\mathrm{d}\nu_{\log}$ on each log scale proved above.

1.B. **Method of auxiliary Monge-Ampère equations**

In this subsection, we review the method of auxiliary Monge-Ampère equations introduced by Chen–Cheng [CC21] and Guo–Phong–Tong, and further developed by Guo–Phong–Song–Sturm in a recent series of papers [GPT23, GPS24, GPSS24a, GPSS23]. This method is highly effective for estimating Green functions on Kähler manifolds.

The core idea is to construct auxiliary complex Monge-Ampère equations that satisfy a priori bounds, such as the uniform L^∞-estimate in Theorem 1.8, and whose solutions can be used to bound solutions of the corresponding Poisson equations; see Lemma 1.9. For a broader discussion of this technique and its other applications, we refer the reader to the survey [GP22].

Our exposition closely follows the framework of Guedj–Tô [GT25], which adapts naturally to our setting. Throughout this subsection, we work under the following assumptions.

Setup 1. Let $\pi\colon X \to \mathbb{D}_s$ be a proper, surjective holomorphic map with connected fibers, where $X_0 := \pi^{-1}(0)$ is the unique singular fiber. Assume that X is equipped with a Kähler form, and that each regular fiber $X_s := \pi^{-1}(s)$, for $s \neq 0$, is an n-dimensional complex manifold.

Compared with Setting 1.5 in [GT25], we drop the assumption that X_0 is irreducible, thus allowing an arbitrary singular central fiber.

In this subsection, we fix a semi-positive form β on X, set $\beta_s := \beta|_{X_s}$, and assume that $V_{\beta_s} := \int_{X_s} \beta_s^n$ is uniformly bounded away from 0 and ∞. Let $\mathrm{d}V_{\beta_s} := \frac{\beta_s^n}{V_{\beta_s}}$ be the normalized volume form on X_s.

Following [GT25, Definition 1.6], we introduce the following class. We fix a positive $\delta \in (0, 1]$. Here a relative Kähler form on $\pi^{-1}(\mathbb{D}_\delta) \setminus X_0$ means a smooth real $(1, 1)$-form whose restriction to each fiber X_s, $s \in \mathbb{D}_\delta^\circ$, is a Kähler form.

Definition 1.5. *Fix $p > 1$ and $B, C, \alpha > 0$, $\delta \in (0, 1)$. We let $\mathcal{K}_{\mathrm{Skoda}}((X, \beta), p, B, C, \alpha, \delta)$ denote the set of all relative Kähler forms θ on $\pi^{-1}(\mathbb{D}_\delta) \setminus X_0$ such that:*

(1) For every $s \in \mathbb{D}_\delta^\circ$, the restriction $\theta_s := \theta|_{X_s}$ satisfies the L^p-condition on its volume density, namely
$$\int_{X_s} f_s^p \,\mathrm{d}V_{\beta_s} \le B, \qquad \frac{\theta_s^n}{V_{\theta_s}} = f_s \,\mathrm{d}V_{\beta_s},$$
where $V_{\theta_s} := \int_{X_s} \theta_s^n$.

(2) For every $s \in \mathbb{D}_\delta^\circ$ and every $\varphi_s \in \mathrm{PSH}(X_s, \theta_s)$ with $\sup_{X_s} \varphi_s = 0$, we have the uniform Skoda estimate
$$\int_{X_s} \exp(-\alpha\varphi_s) \,\mathrm{d}V_{\beta_s} \le C.$$

As a consequence of the rescaled Skoda estimate in Proposition 1.1, we obtain the following proposition.

Proposition 1.6. *Let β be the background Kähler metric on X. By Proposition 1.1, the rescaled relative form $\widetilde{\beta}$ on $\pi^{-1}(\mathbb{D}_\delta) \setminus X_0$, defined by*
$$\widetilde{\beta}|_{X_s} := \widetilde{\beta}_s := \frac{\beta_s}{|\log|s||},$$
belongs to $\mathcal{K}_{\mathrm{Skoda}}((X, \beta), p, 1, C, \alpha, \delta)$ for any $p > 1$ and for some $C, \alpha > 0$, $\delta \in (0, 1)$.

Proof. We shrink the disk to $\mathbb{D}_\delta^\circ$ so Proposition 1.1 applies. For each $s \in \mathbb{D}_\delta^\circ$, the volume density f_s of $\widetilde{\beta}_s$ with respect to $\mathrm{d}V_{\beta_s}$ is equal to 1, since

$$\frac{\widetilde{\beta}_s^n}{V_{\widetilde{\beta}_s}} = \frac{\beta_s^n}{V_{\beta_s}} = \mathrm{d}V_{\beta_s}.$$

Hence $\int_{X_s} f_s^p \,\mathrm{d}V_{\beta_s} = 1$ and the condition in Definition 1.5(1) holds for $B = 1$ and any $p > 1$. The Skoda inequality in Definition 1.5(2) follows directly from Proposition 1.1. □

Remark 1.7. Under Setup 1, assume in addition that X_0 is reduced and irreducible. Fix $\delta \in (0,1)$. Let θ be a relative Kähler form on $\pi^{-1}(\mathbb{D}_\delta) \setminus X_0$. If the following two conditions hold:

- there exist $p > 1$ and $B > 0$ such that
$$\int_{X_s} \left(\frac{\theta_s^n}{V_{\theta_s}\,\mathrm{d}V_{\beta_s}} \right)^p \mathrm{d}V_{\beta_s} \leq B$$
for all $s \in \mathbb{D}_\delta^\circ$;
- there exists $A > 0$ such that $[\theta_s] \leq A[\beta_s]$ for all $s \in \mathbb{D}_\delta^\circ$;

then [GT25, Theorem 1.8] yields

$$\theta \in \mathcal{K}_{\mathrm{Skoda}}((X,\beta), p, B, C, \alpha, \delta),$$

where $\alpha = \alpha(n,p,A,B)$ and $C = C(\alpha,n,p,A,B)$.

By [DNGG22, Theorem A], we have the uniform estimate for Monge-Ampère equations.

Theorem 1.8. *Fix $p > 1$ and $B, C, \alpha > 0, \delta \in (0,1)$. Let $\theta \in \mathcal{K}_{\mathrm{Skoda}}((X,\beta), p, B, C, \alpha, \delta)$. Assume that there exists $\varphi_s \in \mathrm{PSH}(X_s,\theta_s) \cap L^\infty(X_s)$, $p' > 1$ and $B' > 0$ independent of $s \in \mathbb{D}_\delta^\circ$ such that*

$$\frac{1}{V_{\theta_s}} (\theta_s + \mathrm{dd}^c \varphi_s)^n = g_s \mathrm{d}V_{\beta_s},$$

with $\int_{X_s} g_s^{p'} \mathrm{d}V_{\beta_s} \leq B'$. Then $\mathrm{Osc}_{X_s}(\varphi_s) \leq L = L(p', B', C, \alpha, n)$.

Proof. Since $\theta \in \mathcal{K}_{\mathrm{Skoda}}((X,\beta), p, B, C, \alpha, \delta)$, it follows directly from [DNGG22, Theorem A]. Using the notations therein, we take

$$X := X_s, \quad \omega := \theta_s, \quad \nu := \mathrm{d}V_{\beta_s}, \quad \mu := g_s \nu.$$

□

The following comparison lemma is the key ingredient in the method of auxiliary Monge-Ampère equations.

Lemma 1.9 ([GT25, Proposition 1.4]). *Suppose that (X^n, ω) is a compact Kähler manifold of complex dimension n. Fix $t > 0$, $p > 1$, and $0 \leq f \in L^{np}(\omega^n)$. Let v be the unique bounded ω-sh function, and let φ be the unique bounded ω-psh function, satisfying*

$$(\omega + \mathrm{dd}^c v) \wedge \omega^{n-1} = e^{tv} f\, \omega^n \quad \textit{and} \quad (\omega + \mathrm{dd}^c \varphi)^n = e^{nt\varphi} f^n\, \omega^n.$$

Then $\varphi \leq v$.

For $\theta \in \mathcal{K}_{\text{Skoda}}((X,\beta),p,B,C,\alpha,\delta)$, we obtain the following Laplacian estimates on (X_s,θ_s) for $s \in \mathbb{D}_\delta^\circ$.

Lemma 1.10 ([GT25, Lemma 2.1]). *Fix $a>0$, and let v be a quasi-subharmonic function on X_s such that $\Delta_{\theta_s} v \geq -a$. Then*

$$\sup_{X_s} v \leq L\left(a + \frac{1}{V_{\theta_s}} \int_{X_s} |v|\ \theta_s^n\right),$$

where $L = L(n,p,B,C,\alpha) > 0$ depends only on n,p,B,C,α.

Proof. The proof of [GT25, Lemma 2.1] uses only the Skoda estimate in Definition 1.5(2), the uniform L^∞-estimate in Theorem 1.8, and the comparison lemma Lemma 1.9. Since these ingredients are available for $\theta \in \mathcal{K}_{\text{Skoda}}(X,p,B,C,\alpha,\delta)$, the same argument applies. □

Proposition 1.11 ([GT25, Proposition 2.2]). *Let u be a continuous function on X_s such that $\int_{X_s} u\, \theta_s^n = 0$, $|\Delta_{\theta_s} u| \leq 1$. Then*

$$\|u\|_{L^\infty(X_s)} \leq L,$$

where $L = L(n,p,B,C,\alpha) > 0$ depends only on n,p,B,C,α.

Proof. The proof of [GT25, Proposition 2.2] relies on the previous lemma together with Theorem 1.8 and Lemma 1.9. Hence the same argument applies in the present setting. □

Remark 1.12. Proposition 1.11 is also proved by Guo–Phong–Song–Sturm in [GPSS24b]. A related uniform Laplacian estimate, under different integrability hypotheses, appears in [NV24, Proposition 2.3], building on the estimates of [GL25].

1.C. Proof of the lower bound of small eigenvalues

We prove the lower bound of small eigenvalues in the following theorem, which is part of our main theorem Theorem 0.2.

Theorem 1.13. *Let $\pi\colon (X,\omega_X) \to \mathbb{D}_s$ be a degeneration of Kähler manifolds of complex dimension n. Suppose X_0 is the unique singular fiber, which may be reducible and non-reduced. Let $\omega_s := \omega_X\,|_{X_s}$ be the restriction of the background Kähler form ω_X on each regular fiber X_s ($s \neq 0$). Then there is a positive constant $C>0$ such that*

$$C\left|\log^{-1}|s|\right| \leq \lambda_1(X_s,\Delta_{\omega_s}),$$

for $0<|s|\ll 1$.

Proof. Let Φ_s be the normalized eigenfunction with the first non-zero eigenvalue of the Laplacian on (X_s,ω_s), i.e.,

$$\Delta_{\omega_s}\Phi_s = \lambda_1(X_s,\Delta_{\omega_s})\Phi_s, \quad \|\Phi_s\|_{L^2(X_s,\omega_s^n)} = 1.$$

By Yoshikawa's spectral convergence theorem (Theorem 0.1), the eigenvalues of (X_s,ω_s) that converge to the spectrum of (Z_{reg},g_Z) are uniformly bounded from above as $s \to 0$. Since the family is smooth away from the singular fiber, after enlarging the constant if necessary, we have $0 \leq \lambda_1(X_s, \Delta_{\omega_s}) \leq \lambda$ for some $0 < \lambda < \infty$ and all $0 < |s| \leq \frac{1}{2}$.

Recall that (X_s, ω_s) satisfies a uniform Sobolev inequality as it constitutes a family of minimal submanifolds (see [Tos10, Lemma 3.2], [DNGG22, Proposition 3.8]). For all $s \neq 0$ and $u \in C^\infty(X_s)$, there is a constant $C_{\mathrm{Sob}} > 0$ independent of s and u such that

$$\begin{aligned} &\left(\int_{X_s} |u|^{2\nu} \omega_s^n\right)^{\frac{1}{\nu}} \leq C_{\mathrm{Sob}}^2 \left(\int_{X_s} |\nabla u|^2_{\omega_s} \omega_s^n + \int_{X_s} |u|^2 \omega_s^n\right), \\ &\left(\nu = \frac{n}{n-1} \quad \text{when } n \geq 2\right); \\ &\left(\int_{X_s} |u|^4 \omega_s^n\right)^{\frac{1}{2}} \leq C_{\mathrm{Sob}}^2 \left(\int_{X_s} |\nabla u|^2_{\omega_s} \omega_s^n + \int_{X_s} |u|^2 \omega_s^n\right), \quad \text{when } n = 1. \end{aligned} \tag{1.4}$$

In the case $n = 1$, we set $\nu = 2$ in the Moser iteration below.

From a standard Moser iteration (see [Pet16, Theorem 9.2.7]), we have

$$V_{\omega_s}^{-1/2} = V_{\omega_s}^{-1/2} \|\Phi_s\|_{L^2(X_s, \omega_s^n)} \leq \|\Phi_s\|_{L^\infty(X_s)} \leq C'$$

for a uniform constant $C' = C'(C_{\mathrm{Sob}}, n, \lambda) > 0$. Indeed, we have

$$\|\Phi_s\|_{L^\infty(X_s)} \leq \exp\left(C_{\mathrm{Sob}} \frac{\sqrt{\nu\lambda}}{\sqrt{\nu} - 1}\right) \|\Phi_s\|_{L^2(X_s, \omega_s^n)}. \tag{1.5}$$

Set $\widetilde{\beta}_s = \frac{\omega_s}{|\log|s||}$. Then $\widetilde{\beta} \in \mathcal{K}_{\mathrm{Skoda}}((X, \omega_X), p, 1, C_{\mathrm{Skoda}}, \alpha_{\mathrm{Skoda}}, \delta)$ for any $p > 1$, for some $0 < \delta < 1$ and for positive constants $C_{\mathrm{Skoda}}, \alpha_{\mathrm{Skoda}}$ in the uniform Skoda inequality by Proposition 1.6.

Note that

$$\Delta_{\widetilde{\beta}_s} \Phi_s = |\log|s|| \, \Delta_{\omega_s} \Phi_s = (|\log|s|| \cdot \lambda_1(X_s, \Delta_{\omega_s})) \, \Phi_s,$$

and

$$\left| \Delta_{\widetilde{\beta}_s} \frac{\Phi_s}{C' \, |\log|s|| \cdot \lambda_1(X_s, \Delta_{\omega_s})} \right| = \left| \frac{\Phi_s}{C'} \right| \leq 1.$$

Applying Proposition 1.11, note that $\int_{X_s} \Phi_s \widetilde{\beta}_s^n = \left|\log^{-n}|s|\right| \int_{X_s} \Phi_s \omega_s^n = 0$, we have

$$\left\| \frac{\Phi_s}{C' \, |\log|s|| \cdot \lambda_1(X_s, \Delta_{\omega_s})} \right\|_{L^\infty(X_s)} \leq C''$$

for $0 < |s| < \delta$ and a uniform $C'' > 0$.

Therefore, we obtain

$$\lambda_1(X_s, \Delta_{\omega_s}) \geq \left\| \frac{\Phi_s}{C'C'' \, |\log|s||} \right\|_{L^\infty(X_s)} \geq C \left|\log^{-1}|s|\right|$$

for a uniform $C = (C'C''V_{\omega_s}^{1/2})^{-1} > 0$, since $V_{\omega_s} = \int_{X_s} \omega_s^n$ is independent of s. □

2. Upper bound

The main theorem of this section is the following.

Theorem 2.1. *Let $\pi\colon \mathcal{Y} \to \mathbb{D}_s$ be a degeneration of compact Kähler manifolds of complex dimension n. Suppose $\mathcal{Y}_0$ is the unique singular fiber and is a reduced divisor with simple normal crossings, i.e., the degeneration is semistable.*

Let β be a closed smooth semi-positive form on $\mathcal{Y}$ such that its restriction to each regular fiber $\mathcal{Y}_s =: \pi^{-1}(s)$ $(s \neq 0)$ is Kähler. Let

$$N_\beta(\mathcal{Y}) := \#\left\{ D \text{ an irreducible component of } \mathcal{Y}_0 \;\middle|\; \int_D \beta^n > 0 \right\}.$$

This number counts only those components of the semistable central fiber carrying positive β-volume. We write $N_\beta := N_\beta(\mathcal{Y})$ and assume $N_\beta \geq 2$.

Then, for the first $N_\beta - 1$ non-zero eigenvalues $0 < \lambda_1(s) \leq \cdots \leq \lambda_{N_\beta - 1}(s)$ of the Laplacian of $(\mathcal{Y}_s, \beta_s =: \beta\,|_{\mathcal{Y}_s})$, there exists a constant $C > 0$ such that

$$\lambda_1(s) \leq \cdots \leq \lambda_{N_\beta - 1}(s) \leq \frac{C}{\log |s|^{-1}}$$

for $0 < |s| \leq \frac{1}{2}$.

In the first subsection, we show that Theorem 2.1 implies the upper bound of small eigenvalues of general degenerations in Theorem 0.3.

In the remaining part of this section, we prove Theorem 2.1 following the ideas in [Yos97] and [DY25, Section 6]. The method is to construct suitable test functions on $\mathcal{Y}_s$ and apply the min-max principle to get the upper bound.

2.A. Proof of the upper bound of small eigenvalues

Assuming Theorem 2.1, we prove the upper bound of small eigenvalues in Theorem 0.3.

We recall the construction of Z in Theorem 0.1. Note that the number of irreducible components of Z minus 1 is equal to the number of small eigenvalues.

Let $\pi\colon (X, \omega_X) \to \mathbb{D}_s$ be a degeneration of Kähler manifolds of complex dimension n. Write the central fiber as $X_0 = \sum_{\alpha=1}^a m_\alpha D_\alpha$, where the D_α are its irreducible components, and let $m = \prod_{\alpha=1}^a m_\alpha$.

Consider the base change and normalization given by the following commutative diagram:

$$(2.1)\qquad \begin{array}{ccccc} \widehat{F^{-1}X} & \xrightarrow{\iota} & F^{-1}X := X \times_{\mathbb{D}_s} \mathbb{D}_t & \xrightarrow{F} & X \\ & \searrow^{\widehat{\Pi}} & \downarrow \Pi & & \downarrow \pi \\ & & \mathbb{D}_t & \xrightarrow{t \mapsto t^m} & \mathbb{D}_s \end{array}$$

We define the reduced divisor $Z := \widehat{\Pi}^{-1}(0)$ as the central fiber of the normalized space.

We next compare the spectral count N_Z with the positive-volume count $N_\beta(\mathcal{Y})$ that appears on a semistable model.

Lemma 2.2. *For any semistable reduction $p\colon \mathcal{Y} \to \mathbb{D}_t$ of π over a finite base change, fitting into the commutative diagram*

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\ \mu\ } & X \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \pi} \\ \mathbb{D}_t & \xrightarrow{\ f_d\ } & \mathbb{D}_s \end{array}$$

where $f_d(t) = t^d = s$, and the induced map $(\mu, p)\colon \mathcal{Y} \to X \times_{\mathbb{D}_s} \mathbb{D}_t$ is an isomorphism over $\mathbb{D}_t^\circ$, set $\beta_{\mathcal{Y}} \coloneqq \mu^*\omega_X$ and define the auxiliary semistable count

$$\begin{aligned} N_{\mathrm{ss}}(X) &\coloneqq N_{\beta_{\mathcal{Y}}}(\mathcal{Y}) \\ &= \#\left\{ D \subseteq \mathcal{Y}_0 \;\middle|\; D \text{ is an irreducible component and } \int_D \beta_{\mathcal{Y}}^n > 0 \right\}. \end{aligned}$$

Then

- $N_{\mathrm{ss}}(X)$ *is independent of the choice of semistable reduction,*
- $N_{\mathrm{ss}}(X) = N_Z$*, where* $N_Z = \#\mathrm{Irr}(Z)$.

Proof. We first show the birational invariance. Let $\mu_i\colon \mathcal{Y}_i \to X$, $i = 1, 2$, be two semistable reductions over the same base change $s = t^d$, and set $\beta_i = \mu_i^*\omega_X$. Let $p_i\colon \mathcal{Y}_i \to \mathbb{D}_t$ be the structure maps.

Since the induced maps $(\mu_i, p_i)\colon \mathcal{Y}_i \to X \times_{\mathbb{D}_s} \mathbb{D}_t$ are isomorphisms over $\mathbb{D}_t^\circ$, the varieties $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are birational over X. We can choose a common log resolution of pairs $(\mathcal{Y}_1, (\mathcal{Y}_1)_0)$ and $(\mathcal{Y}_2, (\mathcal{Y}_2)_0)$ over X, namely, the following diagram:

$$\begin{array}{ccccc} & & \mathcal{W} & & \\ & \swarrow{\scriptstyle \phi_1} & & \searrow{\scriptstyle \phi_2} & \\ \mathcal{Y}_1 & & & & \mathcal{Y}_2 \end{array}$$

Then $\phi_1^*\beta_1 = \phi_2^*\beta_2$. For an irreducible component $G \subset \mathcal{W}_0$, either G is exceptional over $\mathcal{Y}_i$, in which case $\dim \phi_i(G) < n$ and the projection formula gives

$$\int_G \phi_i^*\beta_i^n = 0,$$

or G is the strict transform of a unique irreducible component $D \subset (\mathcal{Y}_i)_0$, in which case

$$\int_G \phi_i^*\beta_i^n = \int_D \beta_i^n.$$

Thus the components with positive β-volume are preserved under passing to a common log resolution. Hence

$$N_{\beta_1}(\mathcal{Y}_1) = N_{\beta_2}(\mathcal{Y}_2)$$

for semistable reductions over the same base change.

The same argument shows invariance under further ramified base change. Indeed, let $p\colon \mathcal{Y} \to \mathbb{D}_t$ be semistable and pull it back by $u \mapsto u^q$. If $p'\colon \mathcal{Y}' \to \mathbb{D}_u$ is a semistable reduction of $\mathcal{Y} \times_{\mathbb{D}_t} \mathbb{D}_u$, and $\psi\colon \mathcal{Y}' \to \mathcal{Y}$ is the induced map, then $\beta_{\mathcal{Y}'} = \psi^*\beta_{\mathcal{Y}}$. Every component of $(\mathcal{Y}')_0$ is either the strict transform of a component of $\mathcal{Y}_0$, or is exceptional over an intersection stratum of $\mathcal{Y}_0$. This follows from the local semistable model: if p

is $z_0 \cdots z_p = t$, then after the base change it is $z_0 \cdots z_p = u^q$, which is smooth at the generic point of each original component and can acquire new divisors only over the strata where at least two components meet. The exceptional components have zero β-volume by the projection formula, while strict transforms have the same volume as the original components. Therefore

$$N_{\beta_{\mathcal{Y}'}}(\mathcal{Y}') = N_{\beta_{\mathcal{Y}}}(\mathcal{Y}).$$

Given two arbitrary semistable reductions of degrees d_1 and d_2, we pass both to the common base change of degree $\ell = \operatorname{lcm}(d_1, d_2)$. The preceding paragraph and the same-base-change invariance then imply that $N_{\text{ss}}(X)$ is independent of the semistable reduction.

It remains to identify this number with the number of irreducible components of Z. Write

$$Z = \widehat{\Pi}^{-1}(0) = \bigcup_{j=1}^{N_Z} Z_j.$$

After a further base change $u \mapsto t = u^q$, take a semistable reduction

$$\nu \colon \mathcal{Y} \to \widehat{F^{-1}X} \times_{\mathbb{D}_t} \mathbb{D}_u.$$

Let $r \colon \mathcal{Y} \to \widehat{F^{-1}X}$ be the induced map and $h := F \circ \iota \colon \widehat{F^{-1}X} \to X$ be the composition of maps appearing in eq. (2.1). Taking $\mu = h \circ r$, we have

$$\beta_{\mathcal{Y}} = \mu^* \omega_X = r^* h^* \omega_X,$$

and we compute $N_{\text{ss}}(X)$ using this semistable model,

$$N_{\text{ss}}(X) = N_{\beta_{\mathcal{Y}}}(\mathcal{Y}).$$

Since $\widehat{F^{-1}X}$ is normal and Z is reduced, the pullback family is smooth at the generic point of each Z_j. Hence the irreducible components of $\mathcal{Y}_0$ are precisely the strict transforms $\widetilde{Z}_j$, together with ν-exceptional divisors. The latter map to subsets of dimension $< n$, so they have zero $\beta_{\mathcal{Y}}$-volume. Since h is finite over the central fiber, for each j, the map

$$h_j := h|_{Z_j} \colon Z_j \to X_0$$

has image an irreducible component $D_{\alpha(j)} \subset X_0$ and is generically finite of degree $e_j \geq 1$. Therefore

$$\int_{\widetilde{Z}_j} \beta_{\mathcal{Y}}^n = e_j \int_{D_{\alpha(j)}} \omega_X^n > 0.$$

Thus the components of $\mathcal{Y}_0$ with positive $\beta_{\mathcal{Y}}$-volume are exactly $\widetilde{Z}_1, \ldots, \widetilde{Z}_{N_Z}$. Consequently

$$N_{\text{ss}}(X) = N_{\beta_{\mathcal{Y}}}(\mathcal{Y}) = N_Z,$$

as claimed. □

Then we can prove the upper bound of the main theorem.

Theorem 2.3. *Let $\pi \colon (X, \omega_X) \to \mathbb{D}_s$ be a degeneration of Kähler manifolds of complex dimension n. Suppose X_0 is the unique singular fiber which may be reducible and non-reduced. Let $N_Z = \#\mathrm{Irr}(Z)$, where Z is the reduced central fiber of the normalized base* *change in Equation (2.1). Then we have $N_Z - 1$ small eigenvalues $0 < \lambda_1(s) \le \cdots \le \lambda_{N_Z-1}(s)$ of the Laplacian on $(X_s, \omega_X|_{X_s})$.*

If $N_Z \ge 2$, then there is a uniform constant $C > 0$ such that

$$\lambda_1(s) \le \cdots \le \lambda_{N_Z-1}(s) \le C\left|\log^{-1}|s|\right|$$

for $0 < |s| \ll 1$.

Proof. We use the semistable diagram in Lemma 2.2,

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\ \mu\ } & X \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \pi} \\ \mathbb{D}_t & \xrightarrow{\ f_d\ } & \mathbb{D}_s \end{array}$$

Set $\beta := \mu^*\omega_X$. Then $(\mathcal{Y}_t, \beta_t)$ is isometric to $(X_{f_d(t)}, \omega_{f_d(t)})$ for $t \ne 0$.

By Theorem 2.1, we have

$$0 < \lambda_1(\mathcal{Y}_t, \Delta_{\beta_t}) \le \cdots \le \lambda_{N_\beta(\mathcal{Y})-1}(\mathcal{Y}_t, \Delta_{\beta_t}) \le \frac{C_1}{\log|t|^{-1}}$$

for some $C_1 > 0$ and $|t| \le \frac{1}{2}$.

By Lemma 2.2, we have $N_\beta(\mathcal{Y}) = N_{\mathrm{ss}}(X) = N_Z$. So

$$0 < \lambda_1(X_{f_d(t)}, \Delta_{\omega_{f_d(t)}}) \le \cdots \le \lambda_{N_Z-1}(X_{f_d(t)}, \Delta_{\omega_{f_d(t)}}) \le \frac{C_1}{\log|t|^{-1}}.$$

Since $s = f_d(t) = t^d$, we have

$$\log|t|^{-1} = \frac{1}{d}\log|s|^{-1}.$$

Hence, after replacing C_1 by $C = dC_1$,

$$\lambda_1(s) \le \cdots \le \lambda_{N_Z-1}(s) \le \frac{C}{\log|s|^{-1}}$$

for $0 < |s| \le \frac{1}{2^d}$. ∎

2.B. **A quantitative retraction à la Dai–Yoshikawa**

We work under the geometric assumptions of Theorem 2.1. Thus $\pi\colon \mathcal{Y} \to \mathbb{D}$ is a semistable degeneration of compact Kähler manifolds of complex dimension n, the central fiber $\mathcal{Y}_0$ is a reduced simple normal crossings divisor, and γ is a fixed Kähler metric on $\mathcal{Y}$. We write

$$\mathcal{Y}_0 = \sum_{i=1}^{a} D_i, \qquad \mathrm{Sing}(\mathcal{Y}_0) = \bigcup_{1\le i<j\le a}(D_i \cap D_j),$$

and

$$\mathcal{Y}_0^{\mathrm{reg}} := \mathcal{Y}_0 \setminus \mathrm{Sing}(\mathcal{Y}_0),$$

and we denote by d_γ the distance induced by γ. For a subset $A \subset \mathcal{Y}$ and $r > 0$, we set

$$B_\gamma(A, r) := \{y \in \mathcal{Y} \mid d_\gamma(y, A) < r\}, \quad \overline{B}_\gamma(A, r) := \{y \in \mathcal{Y} \mid d_\gamma(y, A) \le r\}.$$

Proposition 2.4 (Quantitative retraction). *Fix an integer $\nu \geq n$, and set $\epsilon(s) := 2|s|^{\frac{1}{4\nu}}$. Then, after shrinking $\mathbb{D}$ if necessary, there exists a family of diffeomorphisms*

$$F_s \colon \mathcal{Y}_0^{\mathrm{reg}} \setminus \overline{B}_\gamma(\mathrm{Sing}(\mathcal{Y}_0), \epsilon(s)) \longrightarrow F_s\big(\mathcal{Y}_0^{\mathrm{reg}} \setminus \overline{B}_\gamma(\mathrm{Sing}(\mathcal{Y}_0), \epsilon(s))\big) \subset \mathcal{Y}_s$$

with the following properties.

(1) $F_0 = \mathrm{id}_{\mathcal{Y}_0^{\mathrm{reg}}}$.

(2) For every z in the domain of F_s,

$$d_\gamma(F_s(z), z) \leq K_1 |s|^{\frac{3}{4}}.$$

(3) If β is a smooth differential form on $\mathcal{Y}$, then

$$\|F_s^* \beta_s - \beta_0\|_{L^\infty(\mathcal{Y}_0^{\mathrm{reg}} \setminus \overline{B}_\gamma(\mathrm{Sing}(\mathcal{Y}_0), \epsilon(s)))} \leq K_{2,\beta} |s|^{\frac{1}{2}}.$$

(4) Let $\gamma_s := \gamma|_{\mathcal{Y}_s}$, and let $(F_s)_*\chi$ denote the push-forward of a test function χ by F_s, extended by 0 outside the image of F_s. Then for all $\chi, \chi' \in C_0^\infty(\mathcal{Y}_0^{\mathrm{reg}} \setminus \overline{B}_\gamma(\mathrm{Sing}(\mathcal{Y}_0), \epsilon(s)))$,

$$\Big| \big((F_s)_*\chi, (F_s)_*\chi'\big)_{L^2(\mathcal{Y}_s, \gamma_s)} - (\chi, \chi')_{L^2(\mathcal{Y}_0^{\mathrm{reg}}, \gamma_0)} \Big| \leq K_3 |s|^{\frac{1}{2}} \, \|\chi\|_{L^2(\mathcal{Y}_0^{\mathrm{reg}}, \gamma_0)} \, \|\chi'\|_{L^2(\mathcal{Y}_0^{\mathrm{reg}}, \gamma_0)},$$

and

$$\Big| \big\| d\big((F_s)_*\chi\big) \big\|^2_{L^2(\mathcal{Y}_s, \gamma_s)} - \|d\chi\|^2_{L^2(\mathcal{Y}_0^{\mathrm{reg}}, \gamma_0)} \Big| \leq K_3 |s|^{\frac{1}{2}} \, \|d\chi\|^2_{L^2(\mathcal{Y}_0^{\mathrm{reg}}, \gamma_0)}.$$

In particular,

$$\Big| \big\| d\big((F_s)_*\chi\big) \big\|_{L^2(\mathcal{Y}_s, \gamma_s)} - \|d\chi\|_{L^2(\mathcal{Y}_0^{\mathrm{reg}}, \gamma_0)} \Big| \leq K_3 |s|^{\frac{1}{2}} \, \|d\chi\|_{L^2(\mathcal{Y}_0^{\mathrm{reg}}, \gamma_0)}.$$

(5) Let α be a smooth $(1,1)$-form on $\mathcal{Y}$. Then for every $\chi \in C_0^\infty(\mathcal{Y}_0^{\mathrm{reg}} \setminus \overline{B}_\gamma(\mathrm{Sing}(\mathcal{Y}_0), \epsilon(s)))$,

$$\left| \int_{\mathcal{Y}_s} (F_s)_*\chi \, \alpha_s^n - \int_{\mathcal{Y}_0} \chi \, \alpha_0^n \right| \leq K_{4,\alpha} |s|^{\frac{1}{2}} \, \|\chi\|_{L^1(\mathcal{Y}_0^{\mathrm{reg}}, \gamma_0)},$$

and

$$\left| \int_{\mathcal{Y}_s} d\big((F_s)_*\chi\big) \wedge d^c\big((F_s)_*\chi\big) \wedge \alpha_s^{n-1} - \int_{\mathcal{Y}_0} d\chi \wedge d^c\chi \wedge \alpha_0^{n-1} \right| \leq K_{4,\alpha} |s|^{\frac{1}{2}} \, \|d\chi\|^2_{L^2(\mathcal{Y}_0^{\mathrm{reg}}, \gamma_0)}.$$

Remark 2.5. The construction of the retraction using a vector flow is classical, as seen in Clemens' retraction [Cle69, Cle77] and in [Kod86, Proof of Th. 2.3]. To the author's knowledge, a quantitative version of this retraction first appeared in [DY25, Section 6] for the degeneration of Riemann surfaces. This flow construction was also utilized recently in [CGP21, Section 2] to study complex Monge-Ampère equations.

We construct a family of diffeomorphisms which sends the smooth part of the singular fiber to nearby smooth fibers. The idea is to pick a C^∞ complex vector field v on $\mathcal{Y} \setminus \mathrm{Crit}(\pi) = \mathcal{Y} \setminus \mathrm{Sing}\, \mathcal{Y}_0$ satisfying $\pi_* v = \partial/\partial s$, and use this vector field to flow points on the singular fiber to nearby smooth fibers.

We begin with a direct Łojasiewicz-type estimate, imitating the same estimate in [DY25, Lemma 6.2].

Lemma 2.6. *There exists a constant $c_0 > 0$ such that*

$$\|d\pi(z)\|_\gamma^2 \geq c_0 \, d_\gamma\big(z, \mathrm{Sing}(\mathcal{Y}_0)\big)^{2n}$$

for all $z \in \pi^{-1}(\mathbb{D}_\rho)$, after shrinking $\rho > 0$ if necessary. In particular, for every integer $\nu \geq n$ there exists $c_\nu > 0$ such that

$$\|d\pi(z)\|_\gamma^2 \geq c_\nu \, d_\gamma\big(z, \mathrm{Sing}(\mathcal{Y}_0)\big)^{2\nu}.$$

Proof. Choose finitely many adapted coordinate charts $U_\lambda \Subset V_\lambda$ covering a neighborhood of $\mathcal{Y}_0$, where V_λ carries coordinates $(z_0, \dots, z_n)$ such that

$$\pi = z_0 \cdots z_p$$

for some $0 \leq p \leq n$, and the components of $\mathcal{Y}_0$ meeting V_λ are given by $\{z_0 = 0\}, \dots, \{z_p = 0\}$. On each U_λ the metric γ is uniformly equivalent to the Euclidean metric g_{Euc}, so it suffices to work in these coordinates.

If $p = 0$, then $d\pi = dz_0$ and the desired lower bound is immediate after shrinking the chart. Thus, near $\mathrm{Sing}(\mathcal{Y}_0)$, we may assume $p \geq 1$. In V_λ we have

$$d\pi = \sum_{j=0}^{p} z_0 \cdots \widehat{z_j} \cdots z_p \, dz_j,$$

hence

$$\|d\pi(z)\|_{g_{\mathrm{Euc}}}^2 = \sum_{j=0}^{p} \big| z_0 \cdots \widehat{z_j} \cdots z_p \big|^2.$$

After reordering the coordinates, we may assume

$$|z_0| \leq |z_1| \leq \cdots \leq |z_p|.$$

Since $\{z_0 = z_1 = 0\} \subset \mathrm{Sing}(\mathcal{Y}_0)$, we have

$$d_\gamma\big(z, \mathrm{Sing}(\mathcal{Y}_0)\big)^2 \lesssim |z_0|^2 + |z_1|^2.$$

On the other hand,

$$\|d\pi(z)\|_{g_{\mathrm{Euc}}}^2 \geq |z_1 \cdots z_p|^2 + |z_0 z_2 \cdots z_p|^2 \geq |z_1|^{2p} + |z_0|^{2p} \geq 2^{1-p}\big(|z_0|^2 + |z_1|^2\big)^p.$$

Therefore

$$\|d\pi(z)\|_\gamma^2 \gtrsim d_\gamma\big(z, \mathrm{Sing}(\mathcal{Y}_0)\big)^{2p} \geq d_\gamma\big(z, \mathrm{Sing}(\mathcal{Y}_0)\big)^{2n}$$

whenever z stays in a sufficiently small neighborhood of $\mathrm{Sing}(\mathcal{Y}_0)$, because $p \leq n$ and $d_\gamma(z, \mathrm{Sing}(\mathcal{Y}_0)) \leq 1$ there. Away from a fixed neighborhood of $\mathrm{Sing}(\mathcal{Y}_0)$, the function $\|d\pi\|_\gamma$ has a positive lower bound on $\pi^{-1}(\mathbb{D}_\rho)$ after shrinking ρ, so the same inequality holds globally, possibly with a smaller constant.

The second assertion follows immediately from the first, since $d_\gamma(z, \mathrm{Sing}(\mathcal{Y}_0)) \leq 1$ near $\mathcal{Y}_0$ and $\nu \geq n$. □

Remark 2.7. Because $|\mathrm{d}\pi|^2$ is locally real analytic, an estimate analogous to the one in Lemma 2.6 can be derived from the first Łojasiewicz inequality (cf. [Mal66, p. 62, Theorem 4.1]). It should be noted, however, that the general Łojasiewicz inequality does not yield precise information regarding the exponent ν in Lemma 2.6.

We now introduce the vector field used in the construction of Proposition 2.4. We follow closely the strategy in [DY25, Section 6].

On $\mathcal{Y} \setminus \mathrm{Sing}(\mathcal{Y}_0)$, define the $(1,0)$-vector field

$$\Theta := \frac{(d\pi)^{\sharp_\gamma}}{\|d\pi\|_\gamma^2}.$$

Then $\pi_*\Theta = \partial/\partial s$. We define real vector fields U, V by $U - iV := 2\Theta$. If $s = u + iv$ for $u := \mathrm{Re}\, s$ and $v := \mathrm{Im}\, s$, then

$$\pi_* U = \frac{\partial}{\partial u}, \qquad \pi_* V = \frac{\partial}{\partial v}.$$

Lemma 2.8. *For every $0 < r \leq 1$, there is a constant $C > 0$, independent of r, such that on $\pi^{-1}(\mathbb{D}_\rho) \setminus B_\gamma(\mathrm{Sing}(\mathcal{Y}_0), r)$,*

$$|U|_\gamma + |V|_\gamma \leq C r^{-\nu}, \qquad |\nabla U|_\gamma + |\nabla V|_\gamma \leq C r^{-2\nu}.$$

Here and below, ∇ denotes a fixed smooth connection on $T\mathcal{Y}$, for instance the Levi-Civita connection of γ. Equivalently, in the finitely many adapted charts used above, one may replace ∇ by ordinary coordinate differentiation; the resulting norms are uniformly comparable, so only the constants change.

Proof. Since $|\Theta|_\gamma = \|d\pi\|_\gamma^{-1}$, the first estimate follows from Lemma 2.6. For the derivative estimate, in the adapted charts the coefficients of γ, γ^{-1}, and π, together with their derivatives up to order two, are uniformly bounded. Differentiating the normalized gradient gives $\nabla \|d\pi\|_\gamma^2 = O(\|d\pi\|_\gamma)$, since the Hessian of π and the first derivatives of γ are bounded. Hence $|\nabla\Theta|_\gamma \leq C \|d\pi\|_\gamma^{-2}$. By Lemma 2.6, this is bounded by $Cr^{-2\nu}$ away from $B_\gamma(\mathrm{Sing}(\mathcal{Y}_0), r)$. The estimates for U, V follow because they are the real and imaginary parts of 2Θ. ☐

For $\theta \in [0, 2\pi]$, set $W^\theta := (\cos\theta) U + (\sin\theta) V$. Then

$$\pi_* W^\theta = (\cos\theta)\frac{\partial}{\partial u} + (\sin\theta)\frac{\partial}{\partial v} = e^{i\theta},$$

where we identify $T\mathbb{D}$ with $\mathbb{C}$. Let

$$M_r := Cr^{-\nu}, \qquad N_r := Cr^{-2\nu}, \qquad \delta_r := \frac{r^{2\nu}}{2C}.$$

Then $\delta_r \leq \min\{r/M_r, 1/(2N_r)\}$ for all $0 < r \leq 1$. Choose $r_0 > 0$ sufficiently small so that

$$B_\gamma(\mathrm{Sing}(\mathcal{Y}_0), 2r_0) \subset \pi^{-1}(\mathbb{D}_\rho) \qquad \text{and} \qquad \delta_r < \rho \quad (0 < r < r_0).$$

In what follows we take $0 < r < r_0$. For $z \in \mathcal{Y}_0^{\mathrm{reg}} \setminus \overline{B}_\gamma(\mathrm{Sing}(\mathcal{Y}_0), 2r)$, let $\Phi^\theta(\eta, z)$ be the solution of

$$\frac{d}{d\eta}\Phi^\theta(\eta, z) = W^\theta_{\Phi^\theta(\eta,z)}, \qquad \Phi^\theta(0, z) = z. \tag{2.2}$$

Along this solution, $\frac{d}{d\eta}\pi(\Phi^\theta(\eta, z)) = \pi_* W^\theta = e^{i\theta}$; since $\pi(z) = 0$, we have $\pi(\Phi^\theta(\eta, z)) = \eta e^{i\theta}$. This solution is defined for all $|\eta| \leq \delta_r$. Indeed, let $(-\tau_-, \tau_+)$ be its maximal interval of existence in $\pi^{-1}(\mathbb{D}_\rho) \setminus \mathrm{Sing}(\mathcal{Y}_0)$. We prove the positive-time direction.

Let T_+ be the first time at which the trajectory enters $\overline{B}_\gamma(\mathrm{Sing}(\mathcal{Y}_0), r)$, namely

$$T_+ := \inf\left\{\eta \in [0, \tau_+) \mid d_\gamma\big(\Phi^\theta(\eta, z), \mathrm{Sing}(\mathcal{Y}_0)\big) \leq r\right\},$$

with the convention that $T_+ = +\infty$ if the set is empty. If $T_+ < +\infty$, then by continuity

$$\Phi^\theta(T_+, z) \in \partial B_\gamma(\mathrm{Sing}(\mathcal{Y}_0), r) \subset \pi^{-1}(\mathbb{D}_\rho) \setminus \mathrm{Sing}(\mathcal{Y}_0),$$

so the ODE can be continued past T_+. Hence $T_+ < \tau_+$.

We claim that $T_+ \geq \delta_r$. Otherwise $T_+ < \delta_r < \infty$, and thus $T_+ < \tau_+$. For $0 \leq \eta < T_+$, the trajectory stays in

$$\pi^{-1}(\mathbb{D}_\rho) \setminus B_\gamma(\mathrm{Sing}(\mathcal{Y}_0), r),$$

so $\left|W^\theta\right|_\gamma \leq M_r$. Hence

$$d_\gamma\big(\Phi^\theta(\eta, z), z\big) \leq \int_0^\eta \left|W^\theta_{\Phi^\theta(\sigma,z)}\right|_\gamma \, d\sigma \leq M_r\eta < M_r\delta_r \leq r/2.$$

Letting $\eta \to T_+$, we get $d_\gamma(\Phi^\theta(T_+, z), z) \leq r/2$. Since $d_\gamma(z, \mathrm{Sing}(\mathcal{Y}_0)) > 2r$, the triangle inequality gives

$$d_\gamma\big(\Phi^\theta(T_+, z), \mathrm{Sing}(\mathcal{Y}_0)\big) \geq \frac{3}{2}r,$$

contradicting the definition of T_+. Thus $T_+ \geq \delta_r$.

It follows that the trajectory remains in $\pi^{-1}(\mathbb{D}_\rho) \setminus \overline{B}_\gamma(\mathrm{Sing}(\mathcal{Y}_0), r)$ for $0 \leq \eta < \min\{\tau_+, \delta_r\}$. If $\tau_+ \leq \delta_r$, then its image is contained in the compact subset

$$\pi^{-1}(\overline{\mathbb{D}}_{\delta_r}) \setminus B_\gamma(\mathrm{Sing}(\mathcal{Y}_0), r) \subset \pi^{-1}(\mathbb{D}_\rho) \setminus \mathrm{Sing}(\mathcal{Y}_0),$$

where W^θ is smooth. The ODE continuation theorem therefore extends the solution past τ_+, contradicting maximality. Hence $\tau_+ > \delta_r$. The negative-time direction is identical.

Lemma 2.9. *For $0 < r < r_0$, $\theta \in [0, 2\pi]$, $z \in \mathcal{Y}_0^{\mathrm{reg}} \setminus \overline{B}_\gamma(\mathrm{Sing}(\mathcal{Y}_0), 2r)$, and $|\eta| \leq \delta_r$, one has*

$$\Phi^\theta(\eta, z) \in \mathcal{Y}_{\eta e^{i\theta}} \setminus \overline{B}_\gamma(\mathrm{Sing}(\mathcal{Y}_0), r), \qquad d_\gamma\big(\Phi^\theta(\eta, z), z\big) \leq M_r\,|\eta|\,.$$

Moreover, $\Phi^\theta_\eta : z \mapsto \Phi^\theta(\eta, z)$ is a diffeomorphism from $\mathcal{Y}_0^{\mathrm{reg}} \setminus \overline{B}_\gamma(\mathrm{Sing}(\mathcal{Y}_0), 2r)$ onto its image in $\mathcal{Y}_{\eta e^{i\theta}}$.

Proof. The distance estimate and the exclusion of $\overline{B}_\gamma(\mathrm{Sing}(\mathcal{Y}_0), r)$ were proved above. Moreover,

$$\frac{d}{d\eta}\pi(\Phi^\theta(\eta, z)) = \pi_* W^\theta = e^{i\theta}.$$

Since $\pi(z) = 0$, this gives $\pi(\Phi^\theta(\eta, z)) = \eta e^{i\theta}$. The diffeomorphism statement follows from uniqueness of solutions, with inverse obtained by flowing for time $-\eta$. □

Lemma 2.10 ([DY25, Lemma 6.3])**.** *There exists a constant $K_5 > 0$ such that for every $0 < r < r_0$, every $\theta \in [0, 2\pi]$, every $z \in \mathcal{Y}_0^{\mathrm{reg}} \setminus \overline{B}_\gamma(\mathrm{Sing}(\mathcal{Y}_0), 2r)$, and every $0 \leq \eta \leq \delta_r$,*

$$\left\|D\Phi^\theta_\eta(z) - D\Phi^\theta_0(z)\right\| \leq K_5 N_r \eta.$$

Hence, whenever $0 \leq \eta \leq \min\{\delta_r, r^{4\nu}\}$,

$$\left\|D\Phi^\theta_\eta(z) - I\right\| \leq K_6\eta^{1/2}.$$

Proof. Fix a finite atlas by real coordinates near $\mathcal{Y}_0$, and write $D\Phi^\theta_\eta(z)$ for the real Jacobian matrix of Φ^θ_η in these coordinates. Set $\Xi^\theta(\eta, z) := D\Phi^\theta_\eta(z)$. Differentiating eq. (2.2) with respect to the real z-coordinates gives

$$\frac{d}{d\eta}\Xi^\theta(\eta,z) = \big(\nabla W^\theta\big)_{\Phi^\theta(\eta,z)} \cdot \Xi^\theta(\eta,z),$$

where ∇W^θ is the real Jacobian matrix of the coordinate expression of W^θ, and $\cdot$ is ordinary matrix multiplication. Define

$$\psi^\theta(\eta) := \left\|\Xi^\theta(\eta,z) - \Xi^\theta(0,z)\right\|.$$

Since Φ_0^θ is the identity map, $\Xi^\theta(0,z) = I$. Thus

$$\Xi^\theta(\eta,z) - \Xi^\theta(0,z) = \int_0^\eta \big(\nabla W^\theta\big)_{\Phi^\theta(\sigma,z)} \cdot \Xi^\theta(\sigma,z)\, d\sigma.$$

Along the trajectory we stay outside $\overline{B}_\gamma(\mathrm{Sing}(\mathcal{Y}_0), r)$, so $\left|\nabla W^\theta\right| \le N_r$ by Lemma 2.8. Hence

$$\psi^\theta(\eta) \le N_r \int_0^\eta \left\|\Xi^\theta(\sigma,z)\right\| d\sigma \le N_r \int_0^\eta \psi^\theta(\sigma)\, d\sigma + N_r\eta,$$

where the last inequality uses $\left\|\Xi^\theta(\sigma,z)\right\| \le \psi^\theta(\sigma) + \left\|\Xi^\theta(0,z)\right\|$, and the uniform bound for $\left\|\Xi^\theta(0,z)\right\|$ is absorbed into the constant. By Gronwall's inequality, $\psi^\theta(\eta) \le e^{N_r\eta} - 1 \le 2N_r\eta$, since $N_r\eta \le N_r\delta_r \le 1/2$. This proves the first estimate.

If $0 \le \eta \le \min\{\delta_r, r^{4\nu}\}$, then

$$N_r\eta \le Cr^{-2\nu}\eta = C\eta^{1/2}\big(\eta^{1/2}r^{-2\nu}\big) \le C\eta^{1/2}.$$

This gives the second estimate. ☐

Lemma 2.11 ([DY25, Lemma 6.4])**.** *Let β be a smooth tensor field on $\mathcal{Y}$. There exists a constant $C_\beta > 0$ such that, for every $0 < r < r_0$, every $\theta \in [0,2\pi]$, and every $0 \le \eta \le r^{4\nu}$,*

$$\left\|(\Phi_\eta^\theta)^*\beta_{\eta e^{i\theta}} - \beta_0\right\|_{L^\infty(\mathcal{Y}_0^{\mathrm{reg}}\setminus\overline{B}_\gamma(\mathrm{Sing}(\mathcal{Y}_0),2r))} \le C_\beta\eta^{1/2}.$$

Proof. In finitely many adapted charts, the coefficients of β and their first derivatives are uniformly bounded. By Lemma 2.9, $d_\gamma(\Phi^\theta(\eta,z),z) \le M_r\eta \le C\eta r^{-\nu}$, and by Lemma 2.10, $\left\|D\Phi_\eta^\theta(z) - I\right\| \le C\eta^{1/2}$. After shrinking r_0, this also gives $\left\|(D\Phi_\eta^\theta)^{-1} - I\right\| \le C\eta^{1/2}$, so the same estimate applies to mixed tensors. Thus the coefficient difference between $(\Phi_\eta^\theta)^*\beta_{\eta e^{i\theta}}$ and β_0 is bounded by $O(\eta r^{-\nu}) + O(\eta^{1/2})$. Since $\eta \le r^{4\nu}$, we have $\eta r^{-\nu} \le \eta^{1/2}$. The result follows. ☐

Proof of Proposition 2.4. If $s = 0$, set $F_0 := \mathrm{id}_{\mathcal{Y}_0^{\mathrm{reg}}}$. If $s \neq 0$, write $s = \eta e^{i\theta}$ and set $r := |s|^{1/(4\nu)}$. For $|s|$ sufficiently small so that $r < r_0$ and $\eta = |s| = r^{4\nu} \le \delta_r$, define $F_s := \Phi_\eta^\theta$ on $\mathcal{Y}_0^{\mathrm{reg}} \setminus \overline{B}_\gamma(\mathrm{Sing}(\mathcal{Y}_0), 2r)$, which is exactly $\mathcal{Y}_0^{\mathrm{reg}} \setminus \overline{B}_\gamma(\mathrm{Sing}(\mathcal{Y}_0), \epsilon(s))$.

Property (1) is immediate. By Lemma 2.9,

$$d_\gamma(F_s(z),z) \le M_r\,|s| \le Cr^{-\nu}\,|s| = C\,|s|^{3/4},$$

which proves (2). Property (3) follows from Lemma 2.11. Applying the same lemma to the tensor field γ, we obtain

$$\left\|F_s^*\gamma_s - \gamma_0\right\|_{L^\infty} \le C\,|s|^{1/2}.$$

Since the volume form and the Hodge star operator depend smoothly on the metric,

$$\left\| \frac{F_s^* dv_s}{dv_0} - 1 \right\|_{L^\infty} \le C\,|s|^{1/2}, \qquad \left\| *_{F_s^*\gamma_s} - *_{\gamma_0} \right\|_{L^\infty} \le C\,|s|^{1/2}.$$

The L^∞-norm is $\|\cdot\|_{L^\infty(\mathcal{Y}_0^{\mathrm{reg}} \setminus \overline{B}_\gamma(\mathrm{Sing}(\mathcal{Y}_0),\epsilon(s)))}$. Here the volume form dv_s is γ_s^n.

For $\chi, \chi' \in C_0^\infty(\mathcal{Y}_0^{\mathrm{reg}} \setminus \overline{B}_\gamma(\mathrm{Sing}(\mathcal{Y}_0), \epsilon(s)))$, change of variables gives

$$\big((F_s)_*\chi, (F_s)_*\chi'\big)_{L^2(\mathcal{Y}_s,\gamma_s)} = \int_{\mathcal{Y}_0^{\mathrm{reg}}} \chi\, \overline{\chi'}\, F_s^* dv_s.$$

Hence

$$\left| \big((F_s)_*\chi, (F_s)_*\chi'\big)_{L^2(\mathcal{Y}_s,\gamma_s)} - (\chi, \chi')_{L^2(\mathcal{Y}_0^{\mathrm{reg}},\gamma_0)} \right| \le C\,|s|^{1/2}\, \|\chi\|_{L^2}\, \|\chi'\|_{L^2}.$$

Similarly,

$$\|d((F_s)_*\chi)\|^2_{L^2(\mathcal{Y}_s,\gamma_s)} = \int_{\mathcal{Y}_0^{\mathrm{reg}}} d\chi \wedge *_{F_s^*\gamma_s} \overline{d\chi},$$

and therefore

$$\left| \|d((F_s)_*\chi)\|^2_{L^2(\mathcal{Y}_s,\gamma_s)} - \|d\chi\|^2_{L^2(\mathcal{Y}_0^{\mathrm{reg}},\gamma_0)} \right| \le C\,|s|^{1/2}\, \|d\chi\|^2_{L^2(\mathcal{Y}_0^{\mathrm{reg}},\gamma_0)}.$$

The corresponding estimate for the unsquared norms follows from $|a-b| = |a^2 - b^2|/(a+b) \le |a^2-b^2|/b$ when $b > 0$, and is trivial when $b = 0$. This proves (4).

Finally, for a smooth $(1,1)$-form α, change of variables and Lemma 2.11 applied to α^n give

$$\left| \int_{\mathcal{Y}_s} (F_s)_*\chi\, \alpha_s^n - \int_{\mathcal{Y}_0} \chi\, \alpha_0^n \right| \le C\,|s|^{1/2}\, \|\chi\|_{L^1(\mathcal{Y}_0^{\mathrm{reg}},\gamma_0)},$$

For the energy term, we also compare the complex structures. Let J denote the ambient complex structure. Since J is a smooth tensor, Lemma 2.11 gives

$$\|F_s^*(J|_{T\mathcal{Y}_s}) - J|_{T\mathcal{Y}_0}\|_{L^\infty} \le C\,|s|^{1/2}.$$

Hence, for every test function χ,

$$F_s^*\big(d^c_{\mathcal{Y}_s}((F_s)_*\chi)\big) = d^c_{\mathcal{Y}_0}\chi + E_s(d\chi), \qquad |E_s(d\chi)|_\gamma \le C\,|s|^{1/2}\,|d\chi|_\gamma.$$

Combining this with $F_s^*\alpha_s^{n-1} = \alpha_0^{n-1} + O(|s|^{1/2})$ gives

$$\left| \int_{\mathcal{Y}_s} d((F_s)_*\chi) \wedge d^c((F_s)_*\chi) \wedge \alpha_s^{n-1} - \int_{\mathcal{Y}_0} d\chi \wedge d^c\chi \wedge \alpha_0^{n-1} \right| \le C\,|s|^{1/2}\, \|d\chi\|^2_{L^2(\mathcal{Y}_0^{\mathrm{reg}},\gamma_0)}.$$

This proves (5), and completes the proof. □

2.C. **Test functions on the central fiber**

We work under the geometric conditions in Theorem 2.1. Then the unique singular fiber $\mathcal{Y}_0$ is a reduced snc divisor in $\mathcal{Y}$. We shall construct families of test functions supported on every irreducible component D_i $(1 \le i \le a)$ of $\mathcal{Y}_0$.

Proposition 2.12. *There exist positive constants $c_1, c_2 > 0$ and $\epsilon_0 > 0$ depending on the degeneration $\pi\colon \mathcal{Y} \to \mathbb{D}$ and a Kähler metric γ on $\mathcal{Y}$ such that, for every $0 < \epsilon < \epsilon_0$, there exists a family of smooth functions $\chi_\epsilon^{(i)}$, $1 \le i \le a$, satisfying:*

(1) For each i, the function $\chi_\epsilon^{(i)}$ depends on ϵ continuously.

(2) For $1 \le i \le a$*, we have* $\chi_\epsilon^{(i)} \in C_0^\infty(D_i \setminus \mathrm{Sing}(\mathcal{Y}_0))$*.*
(3) We have $0 \le \chi_\epsilon^{(i)} \le 1$*.*
(4) For $y \in D_i$*, we have* $\chi_\epsilon^{(i)}(y) = 0$ *if* $d_\gamma(y, \mathrm{Sing}(\mathcal{Y}_0)) \le c_1\epsilon$*, and* $\chi_\epsilon^{(i)}(y) = 1$ *if* $d_\gamma(y, \mathrm{Sing}(\mathcal{Y}_0)) \ge c_2\sqrt{\epsilon}$*.*
(5) We have $\left\|\mathrm{d}\chi_\epsilon^{(i)}\right\|^2_{L^2(\mathcal{Y}_0,\gamma)} \le K_4/\log\epsilon^{-1}$*, where* $K_4 > 0$ *is a uniform constant.*

We first define a family of smooth functions on $\mathbb{R}_{>0}$ depending on $0 < \epsilon < 1$. Set

$$u_\epsilon(t) := \frac{\log(t) - \log(\epsilon)}{\log(\sqrt{\epsilon}) - \log(\epsilon)} = \frac{\log(t) - \log(\epsilon)}{-\frac{1}{2}\log(\epsilon)}.$$

Then $u_\epsilon(t) \ge 1$ when $t \ge \sqrt{\epsilon}$, $u_\epsilon(t) \le 0$ when $t \le \epsilon$, and $u_\epsilon'(t) = -2/(t\log\epsilon)$.

Let $\underline{\eta} \in C^\infty(\mathbb{R})$ be a standard bump function such that $0 \le \underline{\eta} \le 1$, $\underline{\eta} = 0$ on $(-\infty, 0]$, and $\underline{\eta} = 1$ on $[1, \infty)$. We define $\varphi_\epsilon(t) := (\underline{\eta} \circ u_\epsilon)(t)$. Then $\varphi_\epsilon = 0$ on $(0, \epsilon)$ and $\varphi_\epsilon = 1$ on $(\sqrt{\epsilon}, \infty)$. Extending it by zero, we regard φ_ϵ as a smooth function on $\mathbb{R}$. Moreover, $\varphi_\epsilon'(t) \ne 0$ only when $\epsilon < t < \sqrt{\epsilon}$, and on this interval,

$$\varphi_\epsilon'(t) = \underline{\eta}'(u_\epsilon(t))\frac{-2}{t\log(\epsilon)}.$$

Thus there exists a constant $C_1 = C_1(\underline{\eta})$ such that $|\varphi_\epsilon'(t)| \le C_1/(t\,|\log\epsilon|)$ for $\epsilon < t < \sqrt{\epsilon}$, and $\varphi_\epsilon'(t) = 0$ elsewhere.

Using the test functions φ_ϵ constructed above, we prove Proposition 2.12 by working on local adapted charts and gluing them with a partition of unity.

Proof of Proposition 2.12. We fix one of the irreducible components D_i of $\mathcal{Y}_0$. We assume it to be D_1 without loss of generality.

For any point $y \in \mathcal{Y}_0$, we construct a triple of adapted coordinate charts around it. Precisely, we have open sets $U_y \Subset V_y \Subset W_y$ containing y such that:

- The coordinate chart $\{z_i\}_{i=0}^n$ of W_y is the polydisc $\mathbb{D}_3^{n+1}$ of radius 3. Under this coordinate chart, $V_y = \mathbb{D}_2^{n+1}$ and $U_y = \mathbb{D}_1^{n+1}$. The point y is the origin.
- The fibration π on W_y is given by $\pi(z_0, z_1, \cdots, z_n) = z_0z_1\cdots z_p$ for some $0 \le p \le n$, where $z_0, \cdots, z_p$ are defining functions of irreducible components of $\mathcal{Y}_0$ intersecting W_y.

Since $\mathcal{Y}_0$ is compact, we can find finitely many such U_y covering $\mathcal{Y}_0$. We denote these open sets by $\{U_\alpha\}_\alpha$.

As $\{V_\alpha\}_\alpha \cup (\mathcal{Y} \setminus \cup_\alpha \overline{U}_\alpha)$ constitutes an open covering of $\mathcal{Y}$, let $\{\eta_\alpha\}_\alpha \cup \{\eta_0\}$ be a partition of unity subordinate to this covering. Then $\sum_\alpha \eta_\alpha = 1$ on $\cup_\alpha U_\alpha$, and each η_α is supported in V_α.

Each chart system $U_\alpha \Subset V_\alpha \Subset W_\alpha$ falls into one of the following cases.

- **Case 1: When $W_\alpha \cap D_1 = \emptyset$.** We define

$$\chi_{\epsilon,\alpha}^{(1)} := \eta_\alpha \cdot 0 = 0. \tag{2.3}$$

  Then $\chi_{\epsilon,\alpha}^{(1)}|_{\mathcal{Y}_0} = 0$ is a smooth function on $\mathcal{Y}_0^{\mathrm{reg}}$.
- **Case 2: When $W_\alpha \cap D_1 \ne \emptyset$ and $W_\alpha \cap \mathrm{Sing}(\mathcal{Y}_0) = \emptyset$.** We define

$$\chi_{\epsilon,\alpha}^{(1)} := \eta_\alpha \cdot 1 = \eta_\alpha. \tag{2.4}$$

Then $\chi^{(1)}_{\epsilon,\alpha}|_{\mathcal{Y}_0}$ is a smooth function on $\mathcal{Y}_0^{\mathrm{reg}}$, since $\mathcal{Y}_0$ is smooth on V_α.

- **Case 3: When $W_\alpha \cap D_1 \neq \emptyset$ and $W_\alpha \cap \mathrm{Sing}(\mathcal{Y}_0) \neq \emptyset$.** After reindexing, we assume that $\{z_0 = 0\} \cap W_\alpha = D_1 \cap W_\alpha$ and that $\pi(z) = z_0 z_1 \cdots z_p$ on W_α for some $1 \leq p \leq n$. Then $\mathrm{Sing}(\mathcal{Y}_0) \cap W_\alpha \subset (\{z_1 = 0\} \cup \cdots \cup \{z_p = 0\}) \cap W_\alpha$ on D_1. We define

$$\chi^{(1)}_{\epsilon,\alpha} := \eta_\alpha \prod_{i=1}^{p} \varphi_\epsilon(|z_i|). \tag{2.5}$$

Then $\chi^{(1)}_{\epsilon,\alpha}$ is a smooth function supported in V_α. By construction of φ_ϵ, it is zero if $|z_i| \leq \epsilon$ for some $1 \leq i \leq p$. Hence it is zero around $\mathrm{Sing}(\mathcal{Y}_0) \cap W_\alpha$, and its restriction to $\mathcal{Y}_0$ is smooth on $\mathcal{Y}_0^{\mathrm{reg}}$.

Finally, we define

$$\chi^{(1)}_{\epsilon} := \sum_\alpha \chi^{(1)}_{\epsilon,\alpha}|_{\mathcal{Y}_0}. \tag{2.6}$$

We now verify the five conditions in Proposition 2.12.

Since φ_ϵ depends on ϵ continuously, so does $\chi^{(1)}_{\epsilon}$. **This verifies (1).**

We verify that $\chi^{(1)}_{\epsilon}$ is supported on D_1. If $y \in \mathcal{Y}_0 \setminus D_1$, then every local contribution vanishes. In **Case 1** this is immediate. In **Case 2**, the set $V_\alpha \cap \mathcal{Y}_0$ lies in the smooth component D_1, so $\eta_\alpha(y) = 0$. In **Case 3**, since $y \in \mathcal{Y}_0 \setminus D_1$, we have $z_i(y) = 0$ for some $1 \leq i \leq p$; thus the product in eq. (2.5) vanishes. Therefore $\chi^{(1)}_{\epsilon} = 0$ on $\mathcal{Y}_0 \setminus D_1$. From the construction, it is also zero around $\mathrm{Sing}(\mathcal{Y}_0)$. Hence $\chi^{(1)}_{\epsilon} \in C^\infty_0(D_1 \setminus \mathrm{Sing}(\mathcal{Y}_0))$. **This verifies (2).**

By construction, every local factor lies between 0 and 1, and $\sum_\alpha \eta_\alpha = 1$ on $\mathcal{Y}_0$. Hence $0 \leq \chi^{(1)}_{\epsilon} \leq 1$ on $\mathcal{Y}_0$. **This verifies (3).**

We now determine the constants for the distance property. Let $0 < L_1 < 1 < L_2$ be constants such that $L_1^2 \omega_{\mathrm{Euc}} \leq \gamma|_{W_\alpha} \leq L_2^2 \omega_{\mathrm{Euc}}$ on every W_α, where $\omega_{\mathrm{Euc}} = \sum_{i=0}^n \sqrt{-1} dz_i \wedge d\bar{z}_i$. We set $c_1 = L_1$ and $c_2 = L_2$.

We choose $0 < r_0 < e^{-1}$ as in Lemma 2.13. Thus, if $y \in D_1$ and $d_\gamma(y, \mathrm{Sing}(\mathcal{Y}_0)) < r_0$, then every V_α with $y \in V_\alpha$ is of **Case 3**. Moreover, for such a chart,

$$\begin{aligned} &L_1 \min_{1\leq i\leq p} |y_i| \leq d_\gamma(y, \mathrm{Sing}(\mathcal{Y}_0)) \quad \text{if } d_\gamma(y, \mathrm{Sing}(\mathcal{Y}_0)) < r_0, \\ &d_\gamma(y, \mathrm{Sing}(\mathcal{Y}_0)) \leq L_2 \min_{1\leq i\leq p} |y_i|. \end{aligned} \tag{2.7}$$

We take $\epsilon_0 = (r_0/c_2)^2$. Since $r_0 < e^{-1}$ and $c_2 > 1$, we have $\epsilon_0 < e^{-2}$. Also, for every $0 < \epsilon < \epsilon_0$, we have $c_1\epsilon < r_0$ and $c_2\sqrt{\epsilon} < r_0$.

We verify the distance property in (4). If $y \in D_1$ and $d_\gamma(y, \mathrm{Sing}(\mathcal{Y}_0)) \leq c_1\epsilon$, then $d_\gamma(y, \mathrm{Sing}(\mathcal{Y}_0)) < r_0$. Hence every nonzero local contribution comes from **Case 3**. By eq. (2.7), $L_1 \min_i |y_i| \leq d_\gamma(y, \mathrm{Sing}(\mathcal{Y}_0)) \leq c_1\epsilon$. Since $c_1 = L_1$, we get $\min_i |y_i| \leq \epsilon$, and so $\chi^{(1)}_{\epsilon,\alpha}(y) = 0$ for every such α. Therefore $\chi^{(1)}_{\epsilon}(y) = 0$.

Conversely, if $y \in D_1$ and $d_\gamma(y, \mathrm{Sing}(\mathcal{Y}_0)) \geq c_2\sqrt{\epsilon}$, then **Case 1** does not contribute. In **Case 2**, the local factor is identically 1. In **Case 3**, eq. (2.7) gives $L_2 \min_i |y_i| \geq d_\gamma(y, \mathrm{Sing}(\mathcal{Y}_0)) \geq c_2\sqrt{\epsilon}$. Since $c_2 = L_2$, we get $\min_i |y_i| \geq \sqrt{\epsilon}$. Thus $\chi^{(1)}_{\epsilon,\alpha}(y) = \eta_\alpha(y)$ in every contributing chart, and summing over α gives $\chi^{(1)}_{\epsilon}(y) = 1$. **This verifies (4).**

We verify the gradient estimate in (5). By the distance property, $\mathrm{d}\chi_\epsilon^{(1)}$ is supported in $A_\epsilon := D_1 \cap \{c_1\epsilon \le d_\gamma(y, \mathrm{Sing}(\mathcal{Y}_0)) \le c_2\sqrt{\epsilon}\}$. Since $c_2\sqrt{\epsilon} < r_0$, only charts of **Case 3** occur on A_ϵ. Hence, on A_ϵ,

$$\mathrm{d}\chi_\epsilon^{(1)} = \sum_{\substack{\alpha \text{ in} \\ \textbf{Case 3}}} \mathrm{d}\left(\eta_\alpha \prod_{i=1}^{p} \varphi_\epsilon(|z_i|)\right).$$

Write $P_{\epsilon,\alpha} := \prod_{i=1}^{p} \varphi_\epsilon(|z_i|)$. Then $\mathrm{d}(\eta_\alpha P_{\epsilon,\alpha}) = P_{\epsilon,\alpha}\mathrm{d}\eta_\alpha + \eta_\alpha \mathrm{d}P_{\epsilon,\alpha}$. By finite multiplicity of the covering,

$$\begin{aligned}\left\|\mathrm{d}\chi_\epsilon^{(1)}\right\|^2_{L^2(\mathcal{Y}_0,\gamma)} &\le C\int_{A_\epsilon} \left|\sum_\alpha P_{\epsilon,\alpha}\mathrm{d}\eta_\alpha\right|^2_\gamma dV_\gamma \\ &\quad + C \sum_{\substack{\alpha \text{ in} \\ \textbf{Case 3}}} \int_{D_1\cap V_\alpha} |\mathrm{d}P_{\epsilon,\alpha}|^2_\gamma \, dV_\gamma.\end{aligned}$$

The first term is bounded by $C\epsilon$, because A_ϵ is contained in the $c_2\sqrt{\epsilon}$-neighborhood of $\mathrm{Sing}(\mathcal{Y}_0) \cap D_1$, this neighborhood has volume $O(\epsilon)$ in D_1, and the functions η_α are fixed.

For the second term, the estimate for φ'_ϵ gives, on each **Case 3** chart,

$$|\mathrm{d}P_{\epsilon,\alpha}|^2_\gamma \le \frac{C}{|\log \epsilon|^2} \sum_{i=1}^{p} \frac{\mathbf{1}_{\{\epsilon < |z_i| < \sqrt{\epsilon}\}}}{|z_i|^2}.$$

Using the equivalence between γ and the Euclidean metric, and integrating over the remaining bounded coordinates, we obtain

$$\int_{D_1\cap V_\alpha} |\mathrm{d}P_{\epsilon,\alpha}|^2_\gamma \, dV_\gamma \le \frac{C}{|\log\epsilon|^2} \sum_{i=1}^{p} \int_\epsilon^{\sqrt{\epsilon}} \frac{dr}{r} \le \frac{C}{\log \epsilon^{-1}}.$$

Since the number of charts is finite and $\epsilon_0 < e^{-2}$, the harmless $C\epsilon$ term is also bounded by $C/\log\epsilon^{-1}$. Therefore

$$\left\|\mathrm{d}\chi_\epsilon^{(1)}\right\|^2_{L^2(\mathcal{Y}_0,\gamma)} \le \frac{K_4}{\log \epsilon^{-1}},$$

where K_4 depends only on the finite adapted covering, the partition of unity, the fixed bump function $\underline{\eta}$, and the metric comparison constants for γ. **This verifies (5).**

The construction for the other irreducible components D_i is the same after reindexing the adapted coordinates. This completes the proof. □

Lastly, we prove the following lemma used above.

Lemma 2.13. *With the finite adapted covering fixed above, there exists $0 < r_0 < e^{-1}$ such that the following holds. If $y \in D_1$ and $d_\gamma(y, \mathrm{Sing}(\mathcal{Y}_0)) < r_0$, then every V_α with $y \in V_\alpha$ is of **Case 3**. Moreover, for every such chart, the distance comparison eq. (2.7) holds.*

Proof. Since charts of **Case 1** do not meet D_1, it suffices to exclude charts of **Case 2**.

Since $V_\alpha \Subset W_\alpha$ and the family of charts is finite, the number

$$\delta_{\mathrm{bd}} := \min_\alpha d_\gamma(\overline{V}_\alpha, \mathcal{Y} \setminus W_\alpha)$$

is positive. Choose $0 < r_0 < \min\{e^{-1}, \delta_{\mathrm{bd}}\}$.

If $y \in D_1$ and $d_\gamma(y, \mathrm{Sing}(\mathcal{Y}_0)) < r_0$, then no chart of **Case 2** can contain y. Indeed, if $y \in V_\alpha$ for some α in **Case 2**, then $\mathrm{Sing}(\mathcal{Y}_0) \subset \mathcal{Y} \setminus W_\alpha$, and therefore

$$d_\gamma(y, \mathrm{Sing}(\mathcal{Y}_0)) \geq d_\gamma(y, \mathcal{Y} \setminus W_\alpha) \geq d_\gamma(\overline{V}_\alpha, \mathcal{Y} \setminus W_\alpha) \geq \delta_{\mathrm{bd}} > r_0,$$

a contradiction. Hence every V_α with $y \in V_\alpha$ is of **Case 3**.

It remains to prove the distance comparison in eq. (2.7). Suppose $y \in D_1 \cap V_\alpha$ and V_α is of **Case 3**. Since $d_\gamma(y, \mathrm{Sing}(\mathcal{Y}_0)) < r_0 \leq \delta_{\mathrm{bd}}$, every closest point of $\mathrm{Sing}(\mathcal{Y}_0)$ to y lies in W_α. Thus the distance is computed inside W_α. In this chart, $D_1 = \{z_0 = 0\}$ and $\mathrm{Sing}(\mathcal{Y}_0) \cap D_1 \cap W_\alpha = \cup_{i=1}^p \{z_0 = z_i = 0\}$. The other local strata of $\mathrm{Sing}(\mathcal{Y}_0)$ are no closer to $y = (0, y_1, \cdots, y_n)$. Therefore the Euclidean distance from y to the local singular locus is $\min_{1\leq i\leq p} |y_i|$.

Since $L_1^2 \omega_{\mathrm{Euc}} \leq \gamma \leq L_2^2 \omega_{\mathrm{Euc}}$ on W_α, the metric distance satisfies $L_1 \min_i |y_i| \leq d_\gamma(y, \mathrm{Sing}(\mathcal{Y}_0))$ whenever $d_\gamma(y, \mathrm{Sing}(\mathcal{Y}_0)) < r_0$. Conversely, joining y to the local singular locus along a coordinate line gives $d_\gamma(y, \mathrm{Sing}(\mathcal{Y}_0)) \leq L_2 \min_i |y_i|$. This proves the lemma. □

Remark 2.14. This construction of test functions is standard; see, for example, [LS05, Section 3].

2.D. Proof of Theorem 2.1

We prove Theorem 2.1 by constructing test functions on $\mathcal{Y}_s$ by flowing test functions on the central fiber (Proposition 2.12) to nearby $\mathcal{Y}_s$ using Proposition 2.4.

We work with the notation of Theorem 2.1, and β is the semi-positive $(1,1)$-form therein. Set $N_\beta := N_\beta(\mathcal{Y})$. We write

$$\mathcal{Y}_0 = \sum_{i=1}^{N_\beta} D_i + \sum_{i=N_\beta+1}^{a} D_i,$$

where $D_1, \cdots, D_{N_\beta}$ are exactly the irreducible components of $\mathcal{Y}_0$ such that $\int_{D_i} \beta^n > 0$.

For each $1 \leq i \leq N_\beta$, with the notation of Proposition 2.12, we define

$$\chi_s^{(i)} := \chi_{2\epsilon(s)/c_1}^{(i)},$$

where $\epsilon(s) = 2|s|^{\frac{1}{4\nu}}$. Since $\chi_s^{(i)} \in C_0^\infty(D_i \setminus \overline{B}_\gamma(\mathrm{Sing}(\mathcal{Y}_0), \epsilon(s)))$ by Proposition 2.12, we define

$$\widetilde{u}_{i,s} := (F_s)_* \chi_s^{(i)} \in C^\infty(\mathcal{Y}_s),$$

where F_s is the diffeomorphism introduced in Proposition 2.4. We shall estimate the L^2-norms of $\widetilde{u}_{i,s}$ and its derivative on $(\mathcal{Y}_s, \beta_s)$.

Using Proposition 2.4(5), we have

$$\int_{\mathcal{Y}_s} \widetilde{u}_{i,s}^2 \beta_s^n = \int_{\mathcal{Y}_0} (\chi_s^{(i)})^2 \beta_0^n + O(|s|^{\frac{1}{2}}) = \int_{D_i} (\chi_s^{(i)})^2 \beta_0^n + O(|s|^{\frac{1}{2}}).$$

From Proposition 2.12(4), $\chi_s^{(i)}$ is 1 on D_i outside a neighborhood of $\mathrm{Sing}(\mathcal{Y}_0)$ of radius $c_2\sqrt{2\epsilon(s)/c_1}$, which has volume $O(\epsilon(s))$ in D_i. Hence

$$\int_{D_i}(\chi_s^{(i)})^2\beta_0^n = \int_{D_i}\beta_0^n - \int_{D_i\cap d_\gamma(y,\mathrm{Sing}(\mathcal{Y}_0))\le c_2\sqrt{2\epsilon(s)/c_1}}(1-(\chi_s^{(i)})^2)\beta_0^n$$
$$= \int_{D_i}\beta_0^n + O(\epsilon(s)).$$

Thus $\|\widetilde{u}_{i,s}\|^2_{L^2(\mathcal{Y}_s,\beta_s)} = \int_{D_i}\beta_0^n + O(|s|^{\frac{1}{4\nu}})$.

For the derivative, using Proposition 2.4(5),

$$\int_{\mathcal{Y}_s}\|\mathrm{d}\widetilde{u}_{i,s}\|^2_{\beta_s}\beta_s^n = n\int_{\mathcal{Y}_0}\mathrm{d}\chi_s^{(i)}\wedge\mathrm{d}^c\chi_s^{(i)}\wedge\beta_0^{n-1} + O(|s|^{\frac{1}{2}}),$$

we recall that β_s is a Kähler form on the regular fiber $\mathcal{Y}_s$.

Using Proposition 2.12(5) and choosing γ to be a background Kähler metric that dominates β, we have

$$n\int_{\mathcal{Y}_0}\mathrm{d}\chi_s^{(i)}\wedge\mathrm{d}^c\chi_s^{(i)}\wedge\beta_0^{n-1} = O(1/\log(\epsilon(s)^{-1})) = O\left(\log^{-1}(|s|^{-1})\right).$$

Thus $\|\mathrm{d}\widetilde{u}_{i,s}\|^2_{L^2(\mathcal{Y}_s,\beta_s)} = O\left(\log^{-1}(|s|^{-1})\right)$.

For each $1\le i\le N_\beta$, we have $\int_{D_i}\beta_0^n>0$. So $\|\widetilde{u}_{i,s}\|^2_{L^2(\mathcal{Y}_s,\beta_s)} = \int_{D_i}\beta_0^n + O(|s|^{\frac{1}{4\nu}})>0$ for s small enough. Then we define

$$u_{i,s} = \frac{\widetilde{u}_{i,s}}{\|\widetilde{u}_{i,s}\|_{L^2(\mathcal{Y}_s,\beta_s)}}.$$

Note that $u_{i,s}$ is supported on $F_s(D_i\backslash\overline{B}_\gamma(\mathrm{Sing}(\mathcal{Y}_0),\epsilon(s)))$. These supports are mutually disjoint. Thus $\{u_{i,s}\}_{i=1}^{N_\beta}$ is an orthonormal system in $L^2(\mathcal{Y}_s,\beta_s)$.

By the min-max principle, for $\lambda_k(s)$ the k-th non-zero eigenvalue of the Laplacian on $(\mathcal{Y}_s,\beta_s)$, we have

$$\lambda_k(s) = \min_{\substack{V\subset C^\infty(\mathcal{Y}_s)\\ \dim V=k+1}}\ \max_{u\in V,\|u\|=1}\|\mathrm{d}u\|^2_{L^2(\mathcal{Y}_s,\beta_s)}.$$

For $1\le k\le N_\beta-1$, we let the $k+1$-dimensional vector space V appearing in the min-max formula be the space spanned by $u_{1,s},\cdots,u_{k+1,s}$. Note that

$$\|\mathrm{d}u_{i,s}\|^2_{L^2(\mathcal{Y}_s,\beta_s)} = \frac{O\left(\log^{-1}(|s|^{-1})\right)}{\int_{D_i}\beta_0^n + O(|s|^{\frac{1}{4\nu}})} = O\left(\log^{-1}(|s|^{-1})\right),\quad 1\le i\le N_\beta.$$

The supports of $u_{1,s},\cdots,u_{k+1,s}$ are mutually disjoint. Thus, for every $u=\sum_{i=1}^{k+1}a_iu_{i,s}\in V$,

$$\|u\|^2_{L^2(\mathcal{Y}_s,\beta_s)} = \sum_{i=1}^{k+1}|a_i|^2,\qquad \|\mathrm{d}u\|^2_{L^2(\mathcal{Y}_s,\beta_s)} = \sum_{i=1}^{k+1}|a_i|^2\|\mathrm{d}u_{i,s}\|^2_{L^2(\mathcal{Y}_s,\beta_s)}.$$

We have

$$\lambda_k(s) = O\left(\log^{-1}(|s|^{-1})\right)$$

for $1\le k\le N_\beta-1$. This completes the proof of Theorem 2.1.

3. Applications and Discussion

3.A. Applications in geometric analysis

Let $\pi\colon (X,\omega_X)\to \mathbb{D}_s$ be a degeneration of Kähler manifolds of complex dimension n. Assume X_0 is the unique singular fiber. Let $\omega_s =: \omega_X|_{X_s}$ denote the restricted metric on the smooth fibers X_s.

When X_0 is reduced and irreducible, the first non-zero eigenvalue of the Laplacian on (X_s,ω_s) is uniformly bounded away from 0 as $s\to 0$. Consequently, there exists a uniform constant $C>0$ such that the following L^2-Poincaré inequality holds for all $s\in\mathbb{D}^\circ_{\frac{1}{2}}$ and all $f\in C^\infty(X_s)$:

$$\left\| f-\frac{\int_{X_s} f\omega_s^n}{\int_{X_s}\omega_s^n}\right\|_{L^2(X_s,\omega_s)} \le C\left\|\mathrm{d}f\right\|_{L^2(X_s,\omega_s)}. \tag{3.1}$$

This uniform inequality is widely utilized in geometric settings involving the degeneration of complex manifolds (see, for example, [RZ11b, Proposition 3.2], [CGP21, Appendix], [DNGG22, Proposition 3.10], [Pan22, Proposition 2.2] and [FT23, Theorem 3.2.4]).

When X_0 is not irreducible, it is well known that the uniform Poincaré constant in Equation (3.1) blows up as $s\to 0$. However, Theorem 0.4 ensures that this rate of blow-up is strictly controlled. This control ultimately yields the following lower bound for the Green function $G_s(x,y)$ on the fibers (X_s,ω_s), which remains valid even for general singular fibers:

Proposition 3.1. *Let $G_s(x,y)$ be the Green function of the Kähler manifold (X_s,ω_s) for $s\neq 0$. Then there exists a constant $C>0$ such that*

$$G_s(x,y)\ge -C\left|\log|s|\right|$$

for all $s\in\mathbb{D}^\circ_{\frac{1}{2}}$.

Proof. Let $\{\Phi_i(s,z)\}_{i\ge 1}$ be eigenfunctions of Δ_{ω_s} with non-zero eigenvalues and normalized L^2-norms. Let k_0 be the number of small eigenvalues; equivalently, $k_0 = N_Z-1$ with the convention that $k_0=0$ if $N_Z=1$. Then we have

$$G_s(x,y)=\underbrace{\sum_{k=1}^{k_0}\frac{1}{\lambda_k(s)}\Phi_k(s,x)\Phi_k(s,y)}_{=:G_{s,\mathrm{low}}(x,y)}+\underbrace{\sum_{k=k_0+1}^{\infty}\frac{1}{\lambda_k(s)}\Phi_k(s,x)\Phi_k(s,y)}_{=:G_{s,\mathrm{high}}(x,y)}.$$

By Yoshikawa's spectral convergence theorem, $\lambda_{k_0+1}(s)$ converges to the first positive eigenvalue of (Z_{reg},g_Z), and hence is uniformly bounded away from 0. Consequently, a classical argument by Cheng and Li [CL81] ensures that $G_{s,\mathrm{high}}(x,y)\ge -C_1$ for some uniform constant $C_1>0$. (See also [Cao26, Proposition 3.6], noting that Theorem 3.7 used therein can be replaced by the higher-dimensional version in [CKS87, Theorem 2.1].)

If $k_0=0$, then $G_{s,\mathrm{low}}=0$. If $k_0>0$, then Theorem 0.4 provides control over the small eigenvalues. Furthermore, uniform Sobolev inequalities combined with Moser iteration bound the L^∞-norms of the corresponding small eigenfunctions. Indeed, from

Equation (1.5), we have

$$\|\Phi_k\|_{L^\infty(X_s)} \leq \exp\left(C_{\mathrm{Sob}}\frac{\sqrt{\nu\lambda}}{\sqrt{\nu}-1}\right).$$

Here $\lambda > 0$ is a uniform bound for small eigenvalues, i.e., $0 < \lambda_1(s) \leq \cdots \leq \lambda_{k_0}(s) \leq \lambda < \infty$ for $s \in \mathbb{D}_{1/2}$, and $\nu = \nu(n)$ is the exponent in the Sobolev inequality.

Together, these bounds yield

$$G_{s,\mathrm{low}}(x,y) \geq -C_2 \left|\log|s|\right|$$

for a uniform constant $C_2 > 0$. This estimate is also trivial when $k_0 = 0$.

Summing these estimates yields the desired bound $G_s(x,y) \geq -C\left|\log|s|\right|$. □

The estimate of the fiberwise Green functions leads to the following two applications.

3.A.1. *Estimates of families of plurisubharmonic functions.*

Proposition 3.2. *Let θ be a smooth $(1,1)$-form on X. Then there is a uniform constant $C_1 > 0$ such that for $s \in \mathbb{D}^\circ_{\frac{1}{2}}$, we have*

$$C_1 \log|s| \leq \frac{1}{\mathrm{vol}(\omega_s)}\int_{X_s} \varphi\omega_s^n - \sup_{X_s}\varphi \leq 0$$

for all $\varphi \in \mathrm{PSH}(X_s,\theta_s)$. Here $\mathrm{vol}(\omega_s) = \int_{X_s}\omega_s^n$ is a positive constant determined by the cohomology class of ω_X.

Proof. Since $\pi^{-1}(\overline{\mathbb{D}_{3/4}})$ is compact, we take a large constant $L > 0$ such that $\theta < L\omega_X$ on $\pi^{-1}(\overline{\mathbb{D}_{3/4}})$. Then for $s \in \mathbb{D}^\circ_{1/2}$, we have $\mathrm{PSH}(X_s,\theta_s) \subset \mathrm{PSH}(X_s, L\omega_s)$. It is enough to prove the estimate for ω_s-plurisubharmonic functions: if $\varphi \in \mathrm{PSH}(X_s,\theta_s)$, then $\varphi/L \in \mathrm{PSH}(X_s,\omega_s)$, and multiplying the resulting estimate by L only changes the constant. Thus we assume below that $\varphi \in \mathrm{PSH}(X_s,\omega_s)$.

Since $\omega_s + dd^c\varphi \geq 0$ for $\varphi \in \mathrm{PSH}(X_s,\omega_s)$, we have

$$-\Delta_{\omega_s}\varphi = \mathrm{tr}_{\omega_s}(dd^c\varphi) = \frac{n\, dd^c\varphi \wedge \omega_s^{n-1}}{\omega_s^n} \geq -n. \tag{3.2}$$

Note that we use the positive definite Laplacian. Thus

$$\frac{1}{\mathrm{vol}(\omega_s)}\int_{X_s}\varphi\omega_s^n - \varphi(z) = \int_{z'\in X_s} G_{\omega_s}(z,z')(-\Delta_{\omega_s}\varphi)(z')\omega_s^n(z').$$

By Proposition 3.1, we have

$$\begin{aligned}
&\int_{z'\in X_s}(G_{\omega_s}(z,z') + C\left|\log|s|\right|)(-\Delta_{\omega_s}\varphi)(z')\omega_s^n(z')\\
&\geq \int_{z'\in X_s}(G_{\omega_s}(z,z') + C\left|\log|s|\right|)(-n)\omega_s^n(z')\\
&\text{(Using } G_{\omega_s}(z,z') + C\left|\log|s|\right| \geq 0 \text{ and eq. (3.2))}\\
&\geq \int_{z'\in X_s} C\left|\log|s|\right|(-n)\omega_s^n(z') = -nC\mathrm{vol}(\omega_s)\left|\log|s|\right|\\
&\text{(Using } \int_{z'\in X_s} G_{\omega_s}(z,z')\omega_s^n(z') = 0\text{)}.
\end{aligned}$$

Since $\int_{z'\in X_s} C\left|\log|s|\right|(-\Delta_{\omega_s}\varphi)\omega_s^n = 0$, we have

$$\frac{1}{\mathrm{vol}(\omega_s)}\int_{X_s}\varphi\omega_s^n - \varphi(z) = \int_{z'\in X_s}(G_{\omega_s}(z,z')+C\left|\log|s|\right|)(-\Delta_{\omega_s}\varphi)(z')\omega_s^n(z') \geq C_1\log|s|.$$

Here $C_1 = n\mathrm{vol}(\omega_s)C$ in the ω_s-plurisubharmonic case; this is independent of s. The general case follows from the preceding scaling reduction.

So we have

$$C_1\log|s| \leq \frac{1}{\mathrm{vol}(\omega_s)}\int_{X_s}\varphi\omega_s^n - \sup_{X_s}\varphi \leq 0.$$

□

Remark 3.3. When the singular fiber X_0 is reduced and irreducible, the $\log|s|$ factor can be removed; see [DNGG22, Conjecture 3.1] and [Ou22, Corollary 4.8].

Remark 3.4. The $\log|s|$ factor is optimal. Indeed, in [DNGG22, Example 3.5], they construct a family of plurisubharmonic functions $\varphi_s \in \mathrm{PSH}(X_s,\omega_s)$ so that $\sup_{X_s}\varphi_s = 0$ and

$$\int_{X_s}\varphi_s\omega_s^n \leq C\log|s|$$

for a constant $C > 0$.

3.A.2. *Estimates of families of Poisson equations.*

Proposition 3.5. *Let g be a continuous function on X and suppose it satisfies the integrability condition on all smooth fibers X_s ($s \neq 0$),*

$$\int_{X_s} g\omega_s^n = 0.$$

Then for $s \in \mathbb{D}^\circ_{\frac{1}{2}}$, let φ_s be the unique solution to the Poisson equation on X_s, i.e.,

$$\Delta_{\omega_s}\varphi_s = g\,|_{X_s},\quad \int_{X_s}\varphi_s\omega_s^n = 0.$$

There exists a constant $C_2 > 0$ independent of g such that

$$\|\varphi_s\|_{L^\infty(X_s)} \leq C_2\left|\log|s|\right|\|g\|_{L^\infty(X_s)}$$

for all $s \in \mathbb{D}^\circ_{\frac{1}{2}}$.

Proof. For $s \in \mathbb{D}^\circ_{\frac{1}{2}}$, we have

$$\varphi_s(z) = \int_{z'\in X_s} G_s(z,z')g(z')\omega_s^n(z').$$

Using $G_{\omega_s}(z,z') + C\left|\log|s|\right| \geq 0$ in Proposition 3.1 and $g + \|g\|_{L^\infty} \geq 0$, we obtain that

$$\begin{aligned}\int_{z'\in X_s} G_s(z,z')g(z')\omega_s^n(z') &= \int_{z'\in X_s}(G_s(z,z')+C\left|\log|s|\right|)g(z')\omega_s^n(z')\\ &\geq \int_{z'\in X_s}(G_s(z,z')+C\left|\log|s|\right|)(-\|g\|_{L^\infty})\omega_s^n(z')\\ &= -C\|g\|_{L^\infty}\mathrm{vol}(\omega_s)\left|\log|s|\right|.\end{aligned}$$

Here $\|g\|_{L^\infty} := \|g\|_{L^\infty(X_s)}$. For the reverse direction, we consider $-g$. This completes the proof. □

Remark 3.6. In the study of holomorphic dynamics, we need to solve fiberwise Poisson equations for degenerating families of Kähler manifolds, e.g. in [FT23, Theorem 3.2.4] and in [Cao26].

Remark 3.7. When we assume the degeneration $\pi\colon X \to \mathbb{D}$ to be semistable and the continuous function g also satisfies the integrability condition on central fibers,

$$\int_D g\omega^n = 0,$$

where D runs over all irreducible components of the singular fiber, then we have better estimates on φ_s, the solution to the fiberwise Poisson equations in Proposition 3.5. Indeed, following the same argument in [Cao26, Theorem 3.5], we have

$$\frac{\|\varphi_s\|_{L^\infty(X_s)}}{|\log|s||^{1/2}} \to 0, \quad s \to 0.$$

3.B. Small eigenvalues of degenerating Kähler-Einstein manifolds

In this subsection, we demonstrate that small eigenvalues also appear for degenerating families of compact Kähler manifolds with canonical metrics.

Example 3.8. Let $\pi\colon X \to \mathbb{D}$ be a degenerating family of Calabi–Yau manifolds, polarized by a relatively ample line bundle L. We equip each smooth fiber X_s ($s \neq 0$) with the unique Ricci-flat Calabi–Yau metric ω_s^{CY} representing the Kähler class $c_1(L)|_{X_s}$. Let $\lambda_k^{\mathrm{CY}}(s)$ denote the k-th non-zero eigenvalue of the Laplacian on $(X_s, \omega_s^{\mathrm{CY}})$.

Since $(X_s, \omega_s^{\mathrm{CY}})$ is a compact Riemannian manifold with non-negative Ricci curvature, classical results of Cheng [Che75, Corollary 2.2] and Wu–Yang–Zhong [Wu91, Theorem 14.2] yield the bounds:

$$\frac{1}{\mathrm{diam}(X_s, \omega_s^{\mathrm{CY}})^2} \le \lambda_k^{\mathrm{CY}}(s) \le \frac{8k^2 n(n+2)}{\mathrm{diam}(X_s, \omega_s^{\mathrm{CY}})^2},$$

where n is the complex dimension of X_s and $\mathrm{diam}(X_s, \omega_s^{\mathrm{CY}})$ is the diameter of $(X_s, \omega_s^{\mathrm{CY}})$.

In light of recent developments in the geometry of degenerating Calabi–Yau manifolds, the asymptotic behavior of $\lambda_k^{\mathrm{CY}}(s)$ depends entirely on the dimension of the essential skeleton $\mathrm{Sk}(X)$ associated to the degeneration (see [KS06, MN15] for the definition). Specifically, we have two cases:

- If $\dim \mathrm{Sk}(X) = 0$, a result of Rong–Zhang [RZ11a, Theorem 1.4] ensures that the diameter is uniformly bounded from above and below; that is, there exists a constant $C > 0$ independent of s such that $C^{-1} \le \mathrm{diam}(X_s, \omega_s^{\mathrm{CY}}) \le C$. Consequently, there exist uniform constants $C_1, C_2 > 0$ such that for all $k \ge 1$,

  $$C_1 \le \lambda_k^{\mathrm{CY}}(s) \le C_2 k^2.$$

  In this case, there are no small eigenvalues.
- If $\dim \mathrm{Sk}(X) \ge 1$, a recent theorem of Li–Tosatti [LT24, Theorem 1.1] establishes that the diameter grows on the order of $|\log|s||^{\frac{1}{2}}$. That is, for a uniform constant

$C > 0$, we have $C^{-1} |\log|s||^{\frac{1}{2}} \le \operatorname{diam}(X_s, \omega_s^{\mathrm{CY}}) \le C |\log|s||^{\frac{1}{2}}$. It then follows that

$$\frac{C_1}{|\log|s||} \le \lambda_k^{\mathrm{CY}}(s) \le \frac{C_2 k^2}{|\log|s||}$$

for uniform constants $C_1, C_2 > 0$ and all $k \ge 1$. In this case, each fixed non-zero eigenvalue tends to 0.

Example 3.9. Let $\pi\colon X \to \mathbb{D}$ be a degenerating family of hyperbolic curves. We equip each smooth fiber X_s ($s \neq 0$) with its unique Kähler–Einstein (hyperbolic) metric $\omega_s^{\mathrm{hyp}} \in c_1(K_{X_s})$. Let $\lambda_k^{\mathrm{hyp}}(s)$ denote the k-th non-zero eigenvalue of the Laplacian on $(X_s, \omega_s^{\mathrm{hyp}})$.

Assume first that the degeneration is stable. By the results of Schoen–Wolpert–Yau [SWY80] and Masur [Mas76] (see also [GHJ01]), there are exactly $N_{\mathrm{hyp}} - 1$ small eigenvalues satisfying the following asymptotic bounds:

$$\frac{C_1}{|\log|s||} \le \lambda_1^{\mathrm{hyp}}(s) \le \cdots \le \lambda_{N_{\mathrm{hyp}}-1}^{\mathrm{hyp}}(s) \le \frac{C_2}{|\log|s||},$$

where N_{hyp} is the number of irreducible components of the central fiber X_0, and $C_1, C_2 > 0$ are constants independent of s.

For a general degeneration of hyperbolic curves, similar asymptotics can be deduced by passing to its stable model via the Deligne–Mumford stable reduction theorem [DM69]. We remark that the base change involved in stable reduction induces isometries on the regular fibers, as the hyperbolic metric is canonically determined by the complex structure.

Remark 3.10. We expect similar eigenvalue asymptotics to hold for degenerations of compact Kähler manifolds of general type equipped with canonical Kähler–Einstein metrics. This would generalize the behavior observed in Example 3.9 to higher dimensions.

Remark 3.11. For a degenerating family of Fano manifolds equipped with canonical Kähler–Einstein metrics normalized by $\operatorname{Ric}(\omega_s) = \omega_s$, there are no small eigenvalues. This follows immediately from the classical theorem of Lichnerowicz, which guarantees a uniform positive lower bound for the first non-zero eigenvalue of the Laplacian.

3.C. **Non-Archimedean picture for small eigenvalues**

We now explain how the appearance of small eigenvalues is reflected in the non-Archimedean limit of a degeneration, in the sense of hybrid convergence of Boucksom–Jonsson.

We first recall the relevant convergence results for complex Monge–Ampère measures. Let $\pi\colon X \to \mathbb{D}$ be a projective degeneration of compact complex manifolds, with central fiber X_0.

- Suppose that X is a Calabi–Yau degeneration polarized by a relatively ample line bundle L. We equip each smooth fiber X_s ($s \neq 0$) with the unique Ricci-flat Calabi–Yau metric $\omega_{\mathrm{CY},s}$ representing $c_1(L)|_{X_s}$. Boucksom–Jonsson [BJ17] proved that the Monge–Ampère measures $\omega_{\mathrm{CY},s}^n$ converge in X^{hyb} to a Lebesgue-type measure on the top-dimensional part of the essential skeleton $\operatorname{Sk}(X)$.

- Suppose that X is a degeneration of compact complex manifolds of general type. We equip each smooth fiber X_s ($s \neq 0$) with its unique Kähler–Einstein metric $\omega_{\mathrm{KE},s} \in c_1(K_{X_s})$. Pille-Schneider [PS22, Theorem A] proved that the Monge–Ampère measures $\omega_{\mathrm{KE},s}^n$ converge in X^{hyb} to a Dirac-type measure supported on the divisorial valuations corresponding to the irreducible components of the central fiber of the canonical model $\mathcal{X}_c$ of X. More precisely,

$$\mu = \sum_{D \in \mathrm{Irr}((\mathcal{X}_c)_0)} ((K_{\mathcal{X}_c})^n \cdot D)\delta_{v_D}.$$

- Let $\omega_{\mathrm{bg},s} := \omega_X|_{X_s}$ be the restriction of a background Kähler metric. Pille-Schneider [PS22, Theorem B] proved that the Monge–Ampère measures $\omega_{\mathrm{bg},s}^n$ converge in X^{hyb} to a Dirac-type measure supported on the divisorial valuations corresponding to the irreducible components of the central fiber of a normal model $\mathcal{X}$ of X with reduced central fiber. More precisely,

$$\mu = \sum_{D \in \mathrm{Irr}(\mathcal{X}_0)} \left(\int_D (\nu^* \omega_X)^n \right) \delta_{v_D},$$

  where $\nu \colon \mathcal{X} \to X$ is the induced map.

We next compare these non-Archimedean limits with the spectral behavior of degenerating curves. We use two elementary spectral models for graphs. First, for a finite connected graph with positive vertex masses, we choose positive edge conductances c_e (for instance $c_e = 1$) and use the weighted graph Laplacian on $\ell^2(V, \mu)$,

$$(\Delta_{\Gamma,\mu} f)(v) = \frac{1}{\mu(v)} \sum_{w \sim v} c_{vw} \big(f(v) - f(w)\big).$$

This is the usual weighted graph Laplacian with vertex measure; see [LLPP15, Section 2, especially (2.3)–(2.7)] with trivial signature. Its Dirichlet energy is

$$\sum_{\{v,w\} \in E} c_{vw} \, |f(v) - f(w)|^2 ,$$

so, since the graph is connected, the kernel consists exactly of the constant functions. Hence, if the graph has N_Γ vertices, the Laplacian has one zero eigenvalue and $N_\Gamma - 1$ positive eigenvalues.

Second, a compact metric graph is a finite graph whose edges are assigned positive lengths, so that each edge is identified with a compact interval. On such a graph, we use the standard Kirchhoff Laplacian: it is $-d^2/dx^2$ on each edge, with continuity of functions and the Kirchhoff condition at every vertex. Its spectrum is discrete and satisfies Weyl's law, so it has infinitely many positive eigenvalues; see [Ber17, Sections 2 and 4.1, especially Lemma 4.4].

- For degenerations of hyperbolic curves with Kähler–Einstein metrics $(X_s, \omega_{\mathrm{KE},s})$, the complex-analytic theory gives exactly $N_{\mathrm{KE}} - 1$ small eigenvalues, each of order $|\log |s||^{-1}$, where N_{KE} is the number of irreducible components of the central fiber of the canonical model $\mathcal{X}_c$; see Example 3.9.

  On the non-Archimedean side, the measures $\omega_{\mathrm{KE},s}$ converge to a Dirac-type measure supported on the divisorial points corresponding to these irreducible components.

Since we are working with a stable model, each component carries strictly positive mass. Thus the dual graph of $(\mathcal{X}_c)_0$, together with the vertex masses $v \mapsto \mu(v)$, becomes a connected weighted graph. Its weighted graph Laplacian has exactly $N_{\mathrm{KE}} - 1$ positive eigenvalues. In the hyperbolic setting, graph-asymptotic results for degenerating Riemann surfaces identify the rescaled small spectrum with the corresponding graph spectrum in appropriate settings; see [GHJ01]. We expect the same spectral convergence picture for higher-dimensional general type degenerations.

- For degenerations of complex curves with induced background metrics $(X_s, \omega_{\mathrm{bg},s})$, we take as normal model the space $\widehat{F^{-1}X}$ introduced in Equation (2.1), and denote its central fiber by Z.

  The complex-analytic results give exactly $N_Z - 1$ small eigenvalues, each of order $|\log|s||^{-1}$, where N_Z is the number of irreducible components of Z; see Theorem 0.4. Meanwhile, the non-Archimedean convergence theorem gives a Dirac-type limiting measure supported on the components of Z. Each of these components has strictly positive mass by Lemma 2.2. Hence the connected dual graph of Z, weighted by $v \mapsto \mu(v)$, again has a weighted graph Laplacian with exactly $N_Z - 1$ positive eigenvalues.

  In the reduced curve case with $N_Z = 2$, the product asymptotic of Dai–Yoshikawa [DY25, Theorem 0.3] implies that the rescaled limit $|\log|s||\ \lambda_1(s)$ exists. We expect that, for general N_Z, the rescaled small eigenvalues converge to the positive eigenvalues of the weighted graph Laplacian of the corresponding dual graph.
- For degenerations of elliptic curves polarized by a relatively ample line bundle L, we equip each smooth fiber X_s $(s \neq 0)$ with the unique Ricci-flat Calabi–Yau metric $\omega_{\mathrm{CY},s}$ representing $c_1(L)|_{X_s}$.

  The complex geometry gives two sharply different possibilities: there are either no small eigenvalues or infinitely many, depending on the dimension of the essential skeleton $\mathrm{Sk}(X)$; see Example 3.8. The non-Archimedean picture gives the same dichotomy. If $\dim \mathrm{Sk}(X) = 0$, then $\mathrm{Sk}(X)$ is a point by connectedness, and the limiting measure is a Dirac measure. The corresponding graph has one vertex, so its weighted graph Laplacian has no positive eigenvalues.

  If $\dim \mathrm{Sk}(X) = 1$, then after passing to a semistable model, the essential skeleton is a metric graph homeomorphic to $\mathbb{S}^1$. The limiting measure is Lebesgue-type on the edges, with positive density. The resulting metric graph Laplacian behaves like the Laplacian on a circle, and therefore has infinitely many positive eigenvalues.

Remark 3.12. When X is a degeneration of compact hyperbolic curves equipped with the Arakelov–Bergman metrics, the corresponding non-Archimedean spectral convergence picture is established in the work of Amini–Nicolussi [AN22]. We also recall that the hybrid limit of the Arakelov–Bergman measures is the Zhang measure, as proved by Shivaprasad [Shi24] and by Amini–Nicolussi [AN25] independently.

In summary, these examples suggest a spectral convergence principle for small eigenvalues under hybrid convergence. On the complex side, the scale $|\log|s||^{-1}$ appears for induced metrics in Theorem 0.4, for Calabi–Yau metrics in Example 3.8, and for hyperbolic metrics in Example 3.9. On the non-Archimedean side, the convergence of the Monge–Ampère measures ω_s^n is known from work of Boucksom–Jonsson in the

Calabi–Yau setting, and from work of Pille-Schneider for induced metrics and for general type degenerations.

In higher dimensions, several substantial pieces of this picture are already available, including non-Archimedean pluripotential theory, hybrid convergence of Monge–Ampère measures, and metric convergence results for special classes of Calabi–Yau degenerations. However, to the best of the author's knowledge, a general canonical spectral object on X^{an}, or on $\mathrm{Sk}(X)$, together with a convergence theorem for the rescaled Dirichlet forms and spectra, is not yet available with the same level of generality as in the one-dimensional theory of Amini–Nicolussi.

Data availability

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

References

[AN22] Omid Amini and Noema Nicolussi, *Moduli of hybrid curves II: Tropical and hybrid Laplacians*, arXiv, 2022. Preprint. arXiv:2203.12785v1.

[AN25] ______, *Moduli of hybrid curves I: Variations of canonical measures*, Annales Scientifiques de l'École Normale Supérieure (March 2025).

[BB90] J. M. Bismut and J. B. Bost, *Fibrés déterminants, métriques de Quillen et dégénérescence des courbes*, Acta Mathematica **165** (1990), 1–103.

[Ber17] Gregory Berkolaiko, *An elementary introduction to quantum graphs*, Geometric and computational spectral theory, 2017, pp. 41–72.

[BJ17] Sébastien Boucksom and Mattias Jonsson, *Tropical and non-Archimedean limits of degenerating families of volume forms*, Journal de l'École polytechnique — Mathématiques **4** (January 2017), 87–139.

[Cao26] Junyu Cao, *The Archimedean height pairing for differential forms on degeneration of Riemann surfaces*, arXiv, 2026. Preprint. arXiv:2603.04802v1.

[CC21] Xiuxiong Chen and Jingrui Cheng, *On the constant scalar curvature Kähler metrics (I)—A priori estimates*, Journal of the American Mathematical Society **34** (June 2021), no. 4, 909–936.

[CGP21] Junyan Cao, Henri Guenancia, and Mihai Păun, *Variation of singular Kähler–Einstein metrics: Kodaira dimension zero (with an appendix by Valentino Tosatti)*, Journal of the European Mathematical Society **25** (December 2021), no. 2, 633–679.

[Che75] Shiu-Yuen Cheng, *Eigenvalue comparison theorems and its geometric applications*, Mathematische Zeitschrift **143** (October 1975), no. 3, 289–297.

[CKS87] E. A. Carlen, S. Kusuoka, and D. W. Stroock, *Upper bounds for symmetric Markov transition functions*, Ann. Inst. Henri Poincaré, Probab. Stat. **23** (1987), 245–287 (English).

[CL81] Shiu-Yuen Cheng and Peter Li, *Heat kernel estimates and lower bound of eigenvalues*, Commentarii Mathematici Helvetici **56** (December 1981), no. 1, 327–338.

[Cle69] C. H. Clemens, *Picard-Lefschetz theorem for families of nonsingular algebraic varieties acquiring ordinary singularities*, Transactions of the American Mathematical Society **136** (1969), 93–93.

[Cle77] ______, *Degeneration of Kähler manifolds*, Duke Mathematical Journal **44** (June 1977), no. 2.

[DM69] Pierre Deligne and David Mumford, *The irreducibility of the space of curves of given genus*, Publications Mathématiques de l'IHÉS **36** (1969), 75–109.

[DNGG22] Eleonora Di Nezza, Vincent Guedj, and Henri Guenancia, *Families of singular Kähler–Einstein metrics*, Journal of the European Mathematical Society **25** (June 2022), no. 7, 2697–2762.

[DY25] Xianzhe Dai and Ken-Ichi Yoshikawa, *Degeneration of Riemann surfaces and small eigenvalues of the Laplacian*, arXiv, 2025. Preprint. arXiv:2509.06151v2.

[FT23] Simion Filip and Valentino Tosatti, *Canonical currents and heights for K3 surfaces*, Cambridge Journal of Mathematics **11** (2023), no. 3, 699–794.

[GHJ01] Joseph F. Grotowski, Jonathan Huntley, and Jay Jorgenson, *Asymptotic behavior of small eigenvalues, short geodesics and period matrices on degenerating hyperbolic Riemann surfaces*, Forum Mathematicum **13** (2001), no. 6, 729–740.

[GL25] Vincent Guedj and Chinh H. Lu, *Quasi-plurisubharmonic envelopes 1: uniform estimates on Kähler manifolds*, Journal of the European Mathematical Society **27** (April 2025), no. 3, 1185–1208.

[GP22] Bin Guo and Duong H. Phong, *Auxiliary Monge–Ampère equations in geometric analysis*, arXiv, 2022. Preprint. arXiv:2210.13308v1.

[GPS24] Bin Guo, Duong H. Phong, and Jacob Sturm, *Green's functions and complex Monge–Ampère equations*, Journal of Differential Geometry **127** (July 2024), no. 3.

[GPSS23] Bin Guo, Duong H. Phong, Jian Song, and Jacob Sturm, *Sobolev inequalities on Kähler spaces*, arXiv, 2023. Preprint. arXiv:2311.00221v1.

[GPSS24a] ______, *Diameter estimates in Kähler geometry*, Communications on Pure and Applied Mathematics **77** (February 2024), no. 8, 3520–3556.

[GPSS24b] ______, *Diameter estimates in Kähler geometry II: removing the small degeneracy assumption*, Mathematische Zeitschrift **308** (October 2024), no. 3.

[GPT23] Bin Guo, Duong H. Phong, and Freid Tong, *On L^∞ estimates for complex Monge-Ampère equations*, Annals of Mathematics **198** (2023), no. 1, 393–418.

[Gro92] Mikhail Gromov, *Spectral geometry of semi-algebraic sets*, Annales de l'Institut Fourier **42** (1992), no. 1-2, 249–274.

[GT25] Vincent Guedj and Tat Dat Tô, *Kähler families of Green's functions*, Journal de l'École polytechnique – Mathématiques **12** (February 2025), 319–339.

[KKMSD73] G. Kempf, F. Knudsen, D. Mumford, and B. Saint-Donat, *Toroidal Embeddings I*, Springer Berlin Heidelberg, 1973.

[Kod86] Kunihiko Kodaira, *Complex Manifolds and Deformation of Complex Structures*, Springer New York, 1986.

[KS06] Maxim Kontsevich and Yan Soibelman, *Affine Structures and Non-Archimedean Analytic Spaces*, The Unity of Mathematics, 2006, pp. 321–385.

[Li24] Yang Li, *Uniform Skoda integrability and Calabi–Yau degeneration*, Analysis & PDE **17** (August 2024), no. 7, 2247–2256.

[LLPP15] Carsten Lange, Shiping Liu, Norbert Peyerimhoff, and Olaf Post, *Frustration index and Cheeger inequalities for discrete and continuous magnetic Laplacians*, Calculus of Variations and Partial Differential Equations **54** (2015), 4165–4196.

[LS05] Zhiqin Lu and Xiaofeng Sun, *On the Weil-Petersson Volume and the First Chern Class of the Moduli Space of Calabi-Yau Manifolds*, Communications in Mathematical Physics **261** (October 2005), no. 2, 297–322.

[LT24] Yang Li and Valentino Tosatti, *Diameter bounds for degenerating Calabi–Yau metrics*, Journal of Differential Geometry **127** (June 2024), no. 2.

[Mal66] Bernard Malgrange, *Ideals of Differentiable Functions*, Tata Institute of Fundamental Research Studies in Mathematics, vol. 3, Oxford University Press, London, 1966.

[Mas76] Howard Masur, *The extension of the Weil-Petersson metric to the boundary of Teichmuller space*, Duke Mathematical Journal **43** (1976), no. 3, 623–635.

[MN15] Mircea Mustaţă and Johannes Nicaise, *Weight functions on non-Archimedean analytic spaces and the Kontsevich–Soibelman skeleton*, Algebraic Geometry (July 2015), 365–404.

[NV24] Duc-Bao Nguyen and Duc-Viet Vu, *Uniform diameter estimates for Kähler metrics in big cohomology classes*, arXiv, 2024. Preprint. arXiv:2410.18532v2.

[Ou22] Wenhao Ou, *Admissible metrics on compact Kähler varieties*, arXiv, 2022. Preprint. arXiv:2201.04821v2.

[Pan22] Chung-Ming Pan, *Singular Gauduchon metrics*, Compositio Mathematica **158** (June 2022), no. 6, 1314–1328.

[Pet16] Peter Petersen, *Riemannian Geometry*, Springer International Publishing, 2016.

[PS22] Léonard Pille-Schneider, *Hybrid convergence of Kähler–Einstein measures*, Annales de l'Institut Fourier **72** (2022July), no. 2, 587–615.

[RZ11a] Xiaochun Rong and Yuguang Zhang, *Continuity of extremal transitions and flops for Calabi-Yau manifolds*, Journal of Differential Geometry **89** (October 2011), no. 2.

[RZ11b] Wei-Dong Ruan and Yuguang Zhang, *Convergence of Calabi–Yau manifolds*, Advances in Mathematics **228** (October 2011), no. 3, 1543–1589.

[Shi24] Sanal Shivaprasad, *Convergence of Bergman measures towards the Zhang measure*, Mathematische Annalen **390** (January 2024), no. 1, 1365–1399.

[SWY80] Richard Schoen, Scott Wolpert, and Shing-Tung Yau, *Geometric bounds on the low eigenvalues of a compact surface*, Geometry of the laplace operator (proc. sympos. pure math., univ. hawaii, honolulu, hawaii, 1979), 1980, pp. 279–285.

[Tos10] Valentino Tosatti, *Adiabatic limits of Ricci-flat Kähler metrics*, Journal of Differential Geometry **84** (February 2010), no. 2.

[Wu91] Hung-Hsi Wu, *The estimate of the first eigenvalue of a compact Riemannian manifold*, Contemporary geometry, 1991, pp. 377–388.

[Yos97] Ken-Ichi Yoshikawa, *Degeneration of algebraic manifolds and the spectrum of Laplacian*, Nagoya Mathematical Journal **146** (June 1997), 83–129.

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